

# NO BV BOUNDS FOR APPROXIMATE SOLUTIONS TO P-SYSTEM WITH GENERAL PRESSURE LAW

ALBERTO BRESSAN, GENG CHEN, QINGTIAN ZHANG, AND SHENGGUO ZHU

ABSTRACT. For the p-system with large BV initial data, an assumption introduced in [3] by Bakhvalov guarantees the global existence of entropy weak solutions with uniformly bounded total variation. The present paper provides a partial converse to this result. Whenever Bakhvalov's condition does not hold, we show that there exist front tracking approximate solutions, with uniformly positive density, whose total variation becomes arbitrarily large. The construction extends the arguments in [4] to a general class of pressure laws.

## 1. INTRODUCTION

A satisfactory existence-uniqueness theory is now available for hyperbolic systems of conservation laws in one space dimension with small total variation [2, 5, 9]. A major remaining open problem is whether the total variation remains uniformly bound or can blow up in finite time for large BV initial data. Up to now, only few systems of hyperbolic conservation laws are known, where uniform BV estimates hold for solutions with large data [12, 15]. On the other hand, examples with finite time blowup have been constructed in [1, 10]. However, these systems do not come from physical models and do not admit a strictly convex entropy.

In this paper, we focus on the p-system with general pressure law modeling barotropic gas dynamics.

$$\begin{cases} u_t + p(v)_x = 0, \\ v_t - u_x = 0, \end{cases} \quad (1.1)$$

where  $v = 1/\rho > 0$  is the specific volume,  $\rho > 0$  is the density and  $u$  is the velocity of the gas. The pressure  $p(v)$  is a smooth function of  $v$  satisfying

$$p_v < 0 \quad \text{and} \quad p_{vv} > 0. \quad (1.2)$$

In [12], Nishida proved the global BV existence to (1.1) with large initial data, for  $\gamma$ -law pressure  $p = v^{-\gamma}$  with  $\gamma = 1$ . On the other hand, in the case  $\gamma = 3$ , various front tracking approximate solutions were recently constructed in [4], exhibiting finite time blowup of the BV norm.

For the p-system with general pressure law, in [3], Bakhvalov extended the global BV existence result for isothermal gas dynamics in [12] to any pressure law  $p(v)$  satisfying the *Bakhvalov's condition*

$$3p_{vv}^2 \leq 2p_v p_{vvv} \quad \text{for all } v > 0. \quad (1.3)$$

In particular, for  $\gamma$ -law pressure  $p = v^{-\gamma}$  with  $\gamma > 0$ , Bakhvalov's condition holds if and only if  $\gamma \in (0, 1]$ . In [3], more general  $2 \times 2$  systems of conservation laws are also considered.

We observe that Bakhvalov's condition determines whether the strength of a shock increases or decreases by crossing a shock of the opposite family, as shown in Figure 1. The shock strength is

here measured by the change of  $h(v)$  across the shock, where

$$h(v) \doteq \int_v^1 \sqrt{-p_v} dv \quad (1.4)$$

is the density part in the Riemann invariants

$$s \doteq u + h(v), \quad r \doteq u - h(v).$$

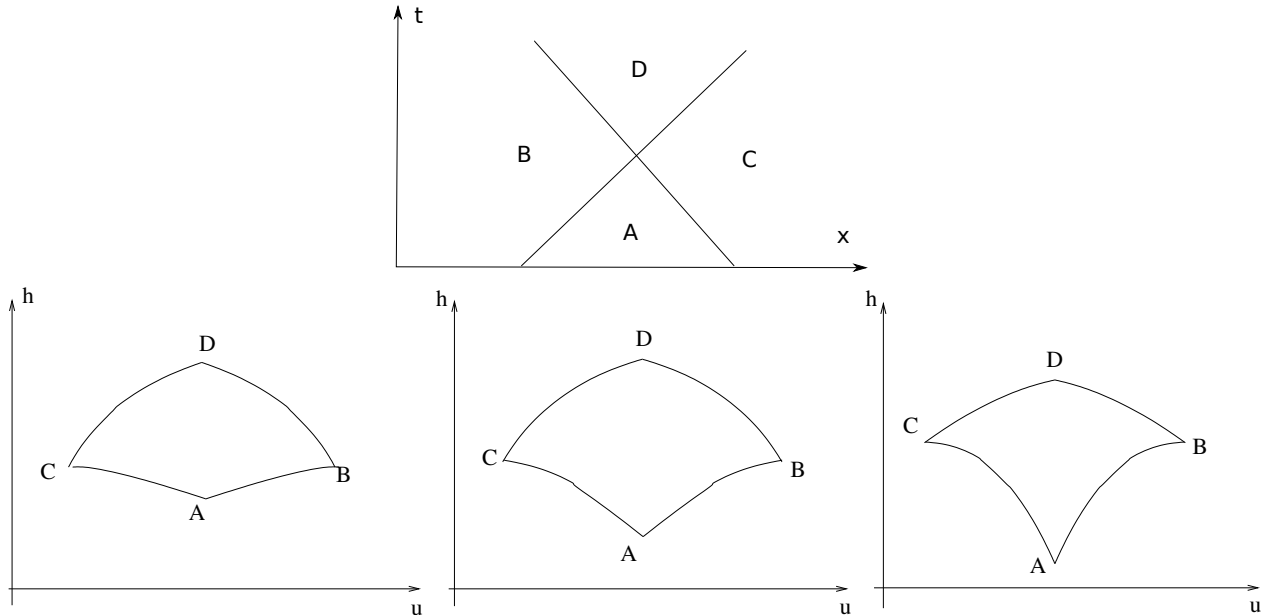


FIGURE 1. The upper figure shows two interacting shocks in the  $x-t$  plane. For this interaction, the lower figures show three different cases, in the  $(u, h)$ -plane Left: the strength is amplified after the crossing, and Bakhvalov's condition (1.3) is not satisfied. Middle: the strength is same after crossing. Right: the strength is reduced after crossing. In the middle and the right pictures, Bakhvalov's condition (1.3) is satisfied. For  $\gamma$ -law pressure with  $\gamma > 0$ , the left figure corresponds to  $\gamma > 1$ , the middle to  $\gamma = 1$ , and the right to  $0 < \gamma < 1$ .

In the present paper we extend the blowup examples in [4] to the case where the pressure violates (1.3). Together with [3], this indicates that Bakhvalov's condition (1.3) is necessary and sufficient for the BV stability of the front tracking scheme. More precisely, the following result will be proved.

**Theorem.** *Assume that the pressure  $p(\cdot)$  satisfies (1.2) for every  $v > 0$  but violates (1.3) for some  $v > 0$ . Then there exists a front tracking approximate solution where the density remains uniformly positive while the total strength of waves approaches infinity as  $t \rightarrow \infty$ . At each wave-front interaction the strengths of outgoing waves are the same as in the exact solution. The only errors introduced by the front tracking approximation are in the speeds of the wave fronts.*

By suitably modifying the construction given in the last section of [4], we expect that one could also construct an example of front tracking approximation where the BV norm blows up in with finite time. The main ideas leading to the blow-up example can be explained as follows.

If Bakhvalov's condition (1.3) fails for  $v$  in a neighborhood of  $v_0$ , one can construct two small approaching shocks such that

- (i) their left and right states remain in the region where Bakhvalov's condition fails, and hence
- (ii) calling  $\sigma_1, \sigma_2$  their sizes before the interaction and  $\sigma'_1, \sigma'_2$  their sizes after the interaction, one has

$$\sigma_1 = \sigma_2 < \sigma'_1 = \sigma'_2. \quad (1.5)$$

Next, assume that these small fronts bounce back and forth between two very large shocks (Fig. 2, left). After a first reflection at the points  $A_1, A_2$ , two rarefactions are created. When these rarefaction impinge again on the large shocks at  $B_1, B_2$ , they generate two new shocks. Every time a front is reflected by a large shock, the outgoing wave is strictly smaller than the incoming one. However, if the shocks  $S_1, S_2$  are very large, the strengths of incoming and reflected fronts are almost the same. Thanks to (1.5), by a suitable choice of the shock strengths, we can achieve

$$\sigma_1 = \sigma_2 = \sigma''_1 = \sigma''_2. \quad (1.6)$$

Hence the interaction pattern can be iterated in time.

If we further increase the strengths of the shocks  $S_1, S_2$ , in (1.6) we would have

$$|\sigma_1| = |\sigma_2| < |\sigma''_1| = |\sigma''_2|. \quad (1.7)$$

To achieve again a periodic pattern, one needs to cancel part of the rarefaction emerging at  $A_2$ . As shown in Fig. 2, right, this can be done by merging it with a shock of the same family, at the interaction point  $A_1$ . In the end, this yields an asymmetric, periodic interaction pattern where

$$\sigma''_1 = \sigma_1, \quad \sigma''_2 = \sigma_2.$$

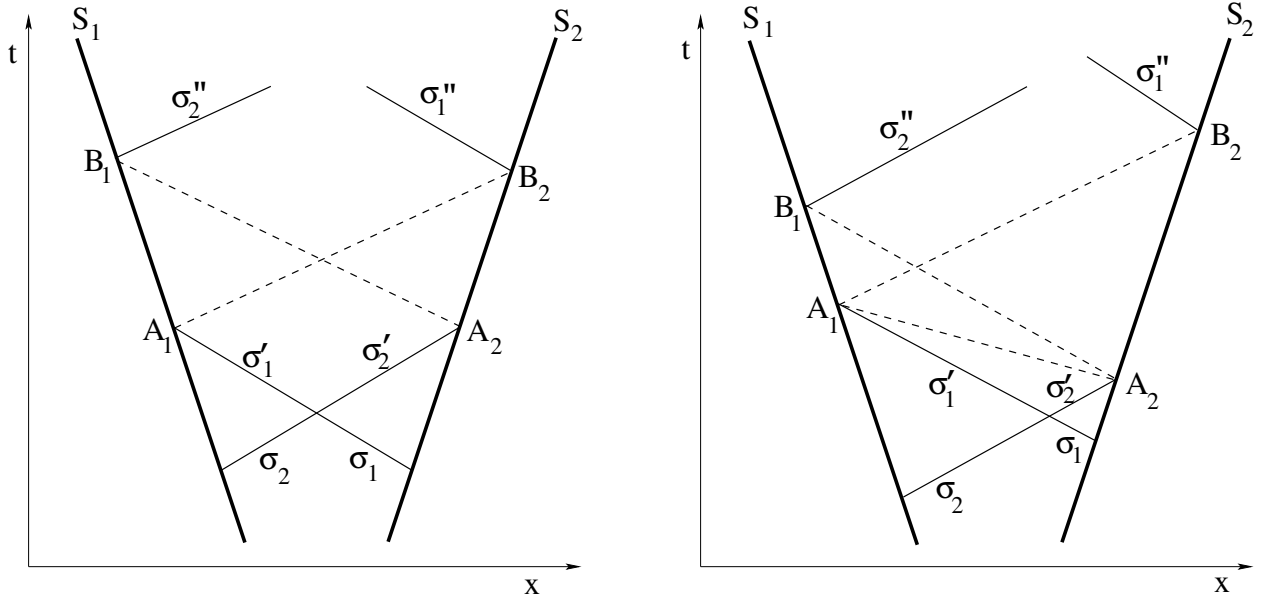


FIGURE 2. Solid lines denote shocks while dotted lines denote rarefaction fronts. Left: a symmetric periodic interaction pattern, where two small fronts bounce back and forth between two large shocks. In the region between the two large shocks, the solution takes values in the region where Bakhvalov's condition fails. Right: an asymmetric interaction pattern. Here part of the rarefaction originating from  $A_2$  is canceled at  $A_1$  by merging with a shock of the same family.

Next, on top of these periodic patterns we add an infinitesimally small wave front (a compression or a rarefaction), as in Fig. 3. If this additional front is initially located at  $P$  and has strength  $\varepsilon$ , after a complete set of interactions we show that

- (i) For the symmetric interaction pattern the strength of the small front at  $Q$  is  $\varepsilon' = \varepsilon + o(\varepsilon)$ .
- (ii) For the asymmetric interaction pattern the strength of the small front at  $Q$  is  $\varepsilon' = \kappa\varepsilon + o(\varepsilon)$ , for some  $\kappa > 1$ .

As this cycle of interactions is repeated over and over, the infinitesimal front is enlarged by an arbitrarily large factor.

Finally, as in [4], we replace this infinitesimally small front with a train of countably many pairs of rarefaction-compression fronts having sizes  $\pm 2^{-k}\varepsilon$ , with  $k = 1, 2, \dots$ . This yields a front-tracking approximate solution satisfying the properties stated in the Theorem (see Fig. 7).

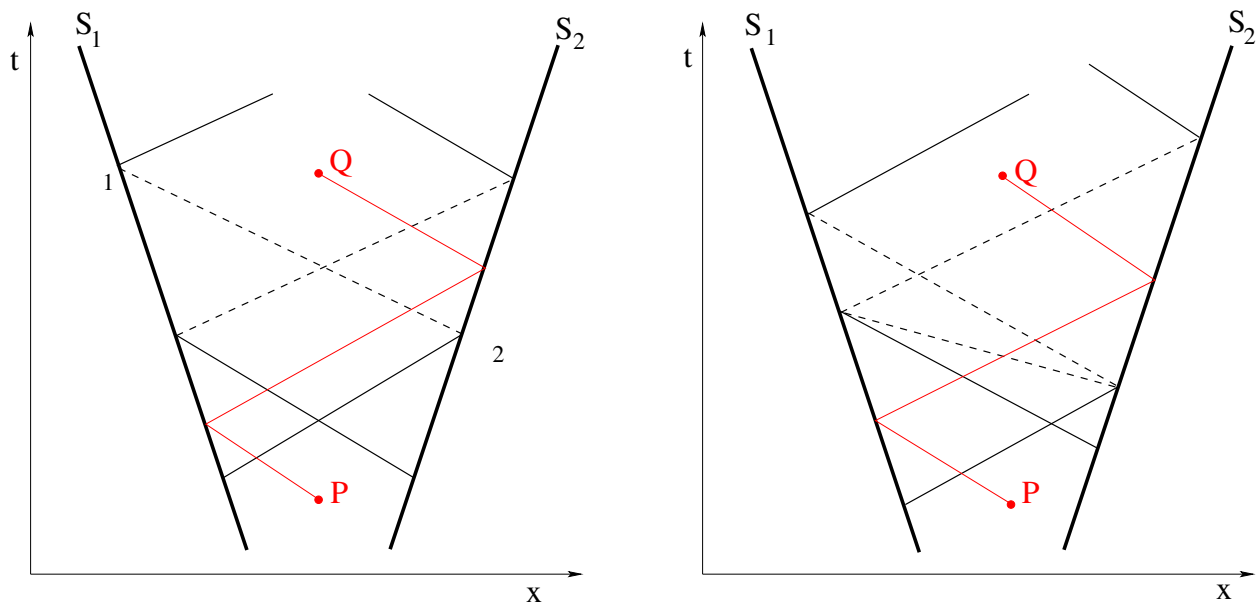


FIGURE 3. If a small front is added on top of the interaction patterns in Fig. 2, after a complete set of interactions the strength of this front is (i) almost the same, in case of the symmetric pattern on the left, and (ii) strictly larger, in case of the asymmetric pattern on the right.

**Remark.** We emphasize that our result does NOT imply that the total variation of entropy weak solutions to the  $p$ -system can become arbitrarily large. Rather, it shows that front tracking approximations can be unstable in the BV norm, whenever Bakhvalov's condition is violated. The present construction also shows that for large initial data, uniform a priori bounds on the total variation cannot be proved simply by estimating the wave strengths at each interaction. As remarked in [4], to establish such BV bounds (if they do indeed hold) it will be essential to use also the decay of rarefaction waves, due to genuine nonlinearity.

The paper is organized as follows. In Sections 2 and 3 we study the wave curves and calculate wave interactions. In Section 4 we first construct a front tracking approximate solution with a periodic interaction pattern. Then, by suitably perturbing this periodic pattern, we give examples of front tracking approximate solutions where the BV-norm blows up as  $t \rightarrow \infty$ .

## 2. WAVE CURVES

In this section, we introduce basic notation and review the rarefaction, compression, and shock curves for (1.1). We omit some standard calculations for wave curves and refer the reader to Chapter 17 in [14] for details.

In Lagrangian coordinates, the wave speed for (1.1) is

$$c = \sqrt{-p'(v)}.$$

Integrating the eigenvectors of (1.1), one obtains the Riemann invariants  $s$  and  $r$ :

$$s \doteq u + h \quad r \doteq u - h, \quad (2.1)$$

with

$$h \equiv h(v) \doteq \int_v^1 \sqrt{-p_v} dv. \quad (2.2)$$

In the case of a smooth solution,  $s$  and  $r$  satisfy

$$s_t + cs_x = 0, \quad r_t - cr_x = 0.$$

This yields the curves for the Rarefaction and Compression simple waves.

For a shock wave, the Rankine-Hugoniot jump conditions take the form

$$\sigma[u] = [p(v)], \quad (2.3)$$

$$\sigma[v] = -[u]. \quad (2.4)$$

where  $[u] = u_r - u_l$ , etc. . . , and the subscripts  $l$  and  $r$  denote the left and right states on the shock wave, respectively. Together with the Lax entropy condition, this uniquely determines the shock curves.

The following table summarizes the equations for rarefaction, compression and shock curves. We refer the reader to Chapter 17 in [14] for detailed calculations. We use  $(\bar{u}, \bar{v})$  and  $(u, v)$  to denote the left and right states across the wave, respectively. Moreover, we use  $\vec{R}$ ,  $\overleftarrow{R}$ ,  $\vec{C}$ ,  $\overleftarrow{C}$ ,  $\vec{S}$ , and  $\overleftarrow{S}$  to denote the forward or backward (or second or first) rarefaction, compression and shock waves respectively.

$$\begin{aligned} \vec{R}: & \quad u - \bar{u} = h(\bar{v}) - h(v), & v > \bar{v} \\ \overleftarrow{R}: & \quad u - \bar{u} = h(v) - h(\bar{v}), & v < \bar{v} \\ \vec{C}: & \quad u - \bar{u} = h(\bar{v}) - h(v), & v < \bar{v} \\ \overleftarrow{C}: & \quad u - \bar{u} = h(v) - h(\bar{v}), & v > \bar{v} \\ \vec{S}: & \quad u - \bar{u} = -\sqrt{(v - \bar{v})(p(\bar{v}) - p(v))}, & v < \bar{v} \\ \overleftarrow{S}: & \quad u - \bar{u} = -\sqrt{(v - \bar{v})(p(\bar{v}) - p(v))}, & v > \bar{v}. \end{aligned} \quad (2.5)$$

We recall that the combined shock-rarefaction curves have  $\mathcal{C}^2$  regularity [2, 14].

## 3. WAVE INTERACTIONS

In this section, we calculate the head-on interactions between two shocks and between a shock and a rarefaction, respectively.

**3.1. Preliminaries.** Consider the function

$$a = a(v, \bar{v}) \doteq h(\bar{v}) - h(v) = \int_{\bar{v}}^v \sqrt{-p_v} dv. \quad (3.1)$$

Since  $a_v(v, \bar{v}) > 0$ , one can recover  $v$  as a function of  $a$  and  $\bar{v}$ , say,  $v = v(a, \bar{v})$ . We also introduce the function

$$F(a, \bar{v}) \doteq \sqrt{(v(a, \bar{v}) - \bar{v})(p(\bar{v}) - p(v(a, \bar{v})))}. \quad (3.2)$$

We now compute the Taylor expansion of  $p(v)$ , for  $v$  near  $\bar{v}$ . In turn, this can be used to calculate the Taylor expansion of  $F(a, \bar{v})$ .

$$p(v) = p(\bar{v}) + p'(\bar{v})(v - \bar{v}) + \frac{1}{2}p''(\bar{v})(v - \bar{v})^2 + \frac{1}{6}p'''(\bar{v})(v - \bar{v})^3 + \frac{1}{24}p^{(4)}(\bar{v})(v - \bar{v})^4 + o(v - \bar{v})^4. \quad (3.3)$$

Using (3.1) and considering  $v = v(a, \bar{v})$ , we compute

$$\begin{aligned} v - \bar{v} &= v(a, \bar{v}) - v(0, \bar{v}) \\ &= (-p'(\bar{v}))^{-\frac{1}{2}}a + \frac{1}{4}(-p'(\bar{v}))^{-2}p''(\bar{v})a^2 + \frac{1}{6}\left[(p''(\bar{v}))^2 - \frac{1}{2}p'(\bar{v})p'''(\bar{v})\right](-p'(\bar{v}))^{-\frac{7}{2}}a^3 + \mathcal{O}(a^4). \end{aligned} \quad (3.4)$$

Using (3.3) and (3.4), we obtain

$$\begin{aligned} F(a, \bar{v}) &= |v - \bar{v}| \cdot \sqrt{\frac{p(\bar{v}) - p(v)}{v - \bar{v}}} \\ &= \text{sign}(v - \bar{v}) \cdot a \{1 + J_1(\bar{v})a^2 + J_2(\bar{v})a^3\} + o(a^4), \end{aligned} \quad (3.5)$$

where

$$J_1 \doteq \frac{1}{96}(-p')^{-3}(p'')^2, \quad (3.6)$$

$$J_2 \doteq \frac{1}{32}p''\left(\frac{1}{2}(p'')^2 - \frac{1}{3}p'''p'\right). \quad (3.7)$$

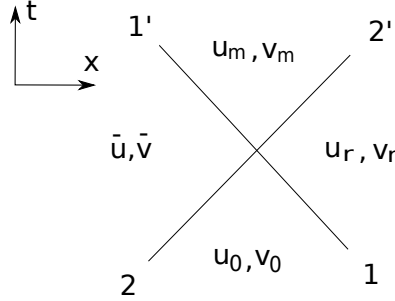


FIGURE 4. Head-on interactions

**3.2. Head-on wave interactions.** In this section, we consider interactions between two opposite waves, as shown in Figure 4, where the incoming waves can be rarefaction, shock, or compression waves. The wave does not change its type after crossing a wave of the opposite family.

We use subscripts 1, 2, 1' and 2' to denote the incoming and outgoing waves of the first and second family, respectively. And we denote the  $(u, v)$  states between these waves according to Figure 4. For any wave-front, we denote by use  $a = h_{\text{left}} - h_{\text{right}}$  the difference between the values of  $h$  at the left and right states of the front. For example, referring to Figure 4, one has

$$a_2 = \bar{h} - h_0, \quad a_1 = h_0 - h_r, \quad a_{1'} = \bar{h} - h_m, \quad a_{2'} = h_m - h_r. \quad (3.8)$$

**Shock-Shock interaction.** In this part, we consider the weak shock-shock interaction. For simplicity, we only consider the case when two shocks have same strength, i.e.  $\bar{v} = v_r$ , hence  $\bar{h} = h_r$ . Using (2.5) and (3.8) with  $\bar{h} = h_r$ , one obtains

$$\begin{cases} a_2 = \bar{h} - h_0 = h_r - h_0 = -a_1 > 0, \\ a_{2'} = h_m - h_r = h_m - \bar{h} = -a_{1'} > 0. \end{cases} \quad (3.9)$$

Then by (2.5) and (3.2), we have

$$F(a_2, \bar{v}) = F(a_{1'}, \bar{v})$$

which yields

$$a_{2'} = -a_{1'} = a_2 \left( 1 + 2J_2(\bar{v})a_2^3 \right) + o(a_2^4), \quad (3.10)$$

where  $J_2$  was defined at (3.7).

**3.3. Rarefaction-Shock or Compression-Shock interaction.** We now consider the interaction between a backward rarefaction and a forward shock. By (2.5), we know  $a_1$ ,  $a_2$ ,  $a_{1'}$ , and  $a_{2'}$  are all negative. Traversing the waves before and after interaction yields

$$-a_{1'} + F(a_{1'}, \bar{v}) = -a_1 + F(a_{1'}, v_0(a_2, \bar{v})).$$

By the equation (3.5), we thus have

$$a_{1'} = a_1 + J_2 a_2 a_1^3 + o(a_2 a_1^3).$$

Hence, by

$$a_1 + a_2 = a_{1'} + a_{2'}$$

we obtain

$$|a_{2'}| = |a_2|(1 + J_2|a_1|^3) + o(|a_2 a_1^3|) \quad (3.11)$$

By an entirely similar calculation, we have same estimate for the interaction between a backward compression and a forward shock. By symmetry, a similar estimate holds for the interaction between a forward compression and a backward shock.

#### 4. FRONT TRACKING APPROXIMATIONS WITH UNBOUNDED BV NORM

In this section, we construct a front tracking approximate solution whose BV norm tends to infinity as  $t \rightarrow \infty$ . We assume that the Bakhvalov condition (1.3) fails at some  $v > 0$ . Hence, by continuity and by (1.2) there exists some interval  $(v_L, v_U)$ , in which

$$J_2 \doteq \frac{1}{32} p''(3p_{vv}^2 - 2p_v p_{vvv}) > 0 \text{ for all } v \in (v_L, v_U). \quad (4.1)$$

**4.1. Front tracking approximations with a periodic interaction pattern.** Following [4], we first construct a symmetric interaction pattern containing four wave fronts, as shown in Fig. 5. This pattern is symmetric, because two boundary shocks  $S_1$  and  $S_2$  (and also the inner shocks  $A_1C$  and  $A_2C$ ) are chosen to have the same strength measured by the difference in  $h$  between two sides of each shock. We choose the strengths of the two large shocks  $S_1, S_2$  and of the two intermediate waves in such a way that, after a whole round of interactions, these strengths are the same as at the initial time. Working in the  $(u, h)$  plane, this is achieved as follows.

- (i) Choose states  $A_1, A_2, B_1, B_2, C$  and  $D$  such that  $v \in (v_L, v_U)$  at these states. Hence (4.1) is satisfied inside and on a neighborhood of the diamond with vertices  $A_1, C, A_2, D$ .
- (ii) Construct two shocks: the 1-shock  $A_1C$  and the 2-shock  $A_2C$ , approaching each other.

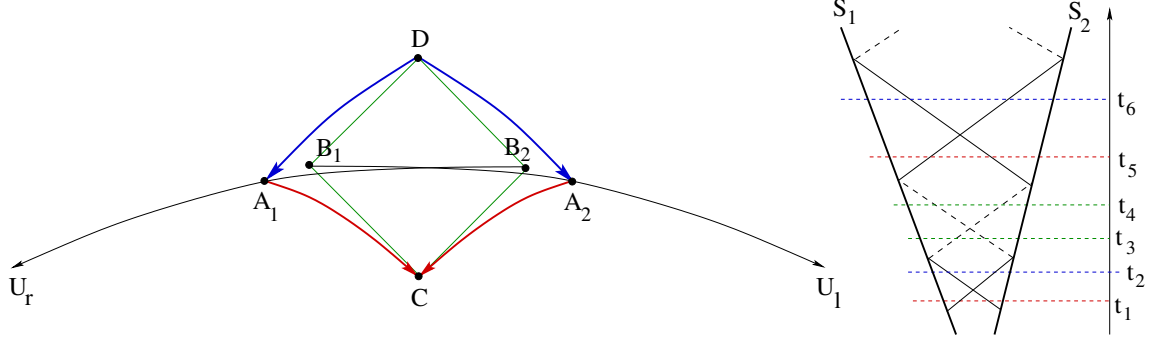


FIGURE 5. A periodic interaction pattern. The left picture is on  $h-u$  plane. The right picture is on  $t-x$  plane

- (iii) Determine the two outgoing shocks  $DA_1$  and  $DA_2$ , resulting from the crossing of the above two shocks.
- (iv) Construct a rectangle having two opposite vertices at  $C$  and  $D$ . Call  $B_1, B_2$  the remaining two vertices.
- (v) Finally, the state  $U_l$  is chosen so that the two points  $B_1$  and  $A_2$  are on the same 1-shock curve with left state  $U_l$ . Symmetrically,  $U_r$  is chosen so that the two points  $B_2$  and  $A_1$  are on the same 2-shock curve with right state  $U_r$ .

We observe that, by (3.10) and (4.1), the  $h$ -component of the states  $B_1$  and  $B_2$  is larger than the  $h$ -component of  $A_1$  and  $A_2$ .

The existence of states  $U_l, U_r$  satisfying (v) is now proved in the following lemma, illustrated in Fig. 6.

**Lemma 4.1.** *In the  $(u, h)$ -plane, consider two points  $B_1 = (u_1, h_1)$  and  $A_2 = (u_2, h_2)$ . Assume that*

- (i)  $u_1 < u_2$ , and  $h_1 > h_2$ .
- (ii) Calling  $A = (u_2, h_2^*)$  the point on the 1-shock curve with right state  $B_1$  with the same  $u$ -component as  $A_2$ , one has  $h_2^* < h_2$ .

*Then there exists a unique  $U_l = (u_l, h_l)$ , with  $0 < h_l < h_2$ , such that both  $B_1$  and  $A_2$  lie on the 1-shock curve with left state state  $U_l$ .*

**Remark 4.2.** Condition (ii) clearly holds when the interaction diamond  $A_1-C-A_2-D$  is small enough, i.e. the interactions inside the diamond are all weak.

*Proof.* We shall use (3.2) with  $(u_l, v_l)$  while  $(u, v) = (u_1, v_1)$  or  $(u_2, v_2)$ . To prove the lemma we need to find  $(u_l, \rho_l)$  such that

$$u_l - u_1 = \sqrt{(p(v_1) - p(v_l))(v_l - v_1)}, \quad u_l - u_2 = \sqrt{(p(v_2) - p(v_l))(v_l - v_2)}. \quad (4.2)$$

This will be achieved if we can find  $v_l$  such that

$$u_2 - u_1 = G(v_l), \quad (4.3)$$

where  $G$  is the function defined as

$$G(v) \doteq \sqrt{(p(v_1) - p(v))(v - v_1)} - \sqrt{(p(v_2) - p(v))(v - v_2)}.$$



The assumption (ii) implies

$$G(v_2) = \sqrt{(p(v_1) - p(v_2))(v_2 - v_1)} < u_2 - u_1.$$

Moreover, a direct computation shows

$$\lim_{v_l \rightarrow \infty} G(v_l) = +\infty.$$

Finally, for any  $v_1 < v_2 < v$ , we have

$$\frac{\partial}{\partial v} G(v) = \frac{-p'(v)(v - v_1) + p(v_1) - p(v)}{2\sqrt{(p(v_1) - p(v))(v - v_1)}} - \frac{-p'(v)(v - v_2) + p(v_2) - p(v)}{2\sqrt{(p(v_2) - p(v))(v - v_2)}} > 0.$$

Indeed, since  $p''(v) > 0$ , one has

$$\frac{\partial}{\partial a} \left( \frac{-p'(v)(v - a) + p(a) - p(v)}{2\sqrt{(p(a) - p(v))(v - a)}} \right) = \frac{\left( p'(a) - \frac{p(v) - p(a)}{v - a} \right) \left( p'(v) - \frac{p(v) - p(a)}{v - a} \right)}{(p(a) - p(v))^3 (v - a)^5} < 0$$

for any  $a < v$ . Since  $v_l \geq v_2$ , there exists a unique value of  $v_l$  such that  $G(v_l) = u_2 - u_1$ . □

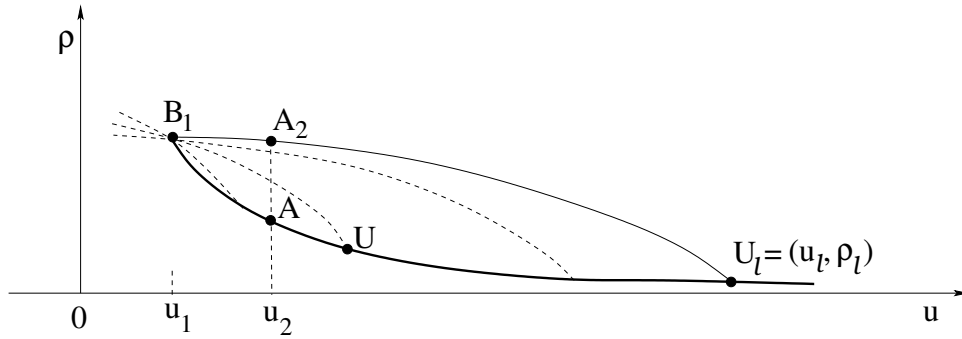


FIGURE 6. By moving the point  $U$  along the 1-shock curve with right state  $B_1$ , we eventually reach a left state  $U_l$  such that the 1-shock curve through  $U_l$  contains  $A_2$  as well.

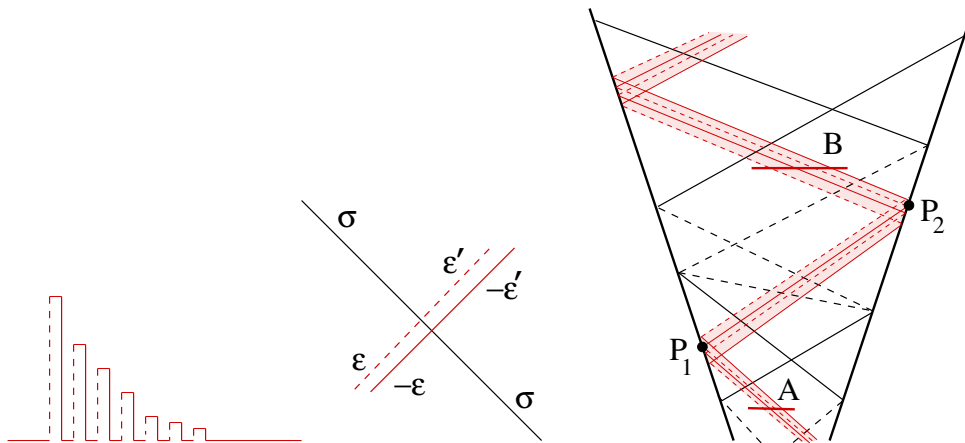


FIGURE 7. A periodic pattern that amplifies a train of small wave fronts.

**4.2. An example with unbounded BV-norm.** Next, as shown in Fig. 7, on top of the periodic pattern constructed in Fig. 5, we add a train of countably many pairs of rarefaction and compression waves. The  $k$ -th pair of waves have sizes  $\pm 2^{-k}\varepsilon$ . Notice that if a front of arbitrary size  $\sigma$  crosses a rarefaction and then a compression wave of exactly opposite sizes, after the two crossings the size of the front is still  $\sigma$ , exactly as before (Fig. 7, center). As a result, the interaction pattern of four large fronts retains its periodicity.

Note that in Fig. 7, we perturb the symmetric periodic pattern in Fig. 5 to an asymmetric periodic pattern by splitting some reflecting rarefaction wave into two pieces. The detail of this perturbation will be discussed later. We recall that the strength of a wave is always defined as

$$|a(v_{\text{left}}, v_{\text{right}})| = |h_{\text{right}} - h_{\text{left}}|,$$

where the subscripts denote the left and right states across the wave-front, respectively.

We always assume that each front in the train of small waves has strength  $\leq \varepsilon$ . Indeed, we can always perform a partial cancellation of the compression-rarefaction pair so that both fronts have strength  $\leq \varepsilon$ . We choose  $\varepsilon > 0$  small enough so that all states between two boundary shocks satisfy  $v \in (v_L, v_U)$ .

We consider the amplification of total wave strength of these alternating waves. To fix the ideas, consider a 1-rarefaction or compression of strength  $\varepsilon_A > 0$ , located at  $A$ . Within a time period, this front will

- i. Cross the intermediate 2-shock.
- ii. Interact with the large 1-shock at  $P_1$  producing a 2-compression.
- iii. Cross the intermediate 1-shock.
- iv. Cross the intermediate 1-rarefaction.
- v. Interact with the large 2-shock at  $P_2$  producing a 2-rarefaction.
- vi. Cross the intermediate 2-rarefaction.

Indeed, when a small wave of strength  $\varepsilon^-$  crosses a shock of the opposite family of strength  $s$ , by (3.11) the strength of the outgoing front is

$$\varepsilon^+ = (1 + J_2 s^3 + o(s^3)) \varepsilon^-. \quad (4.4)$$

When the front crosses a rarefaction of the opposite family, its strength does not change.

Finally, when the small wave impinges on a large shock at  $P_1$  or at  $P_2$ , we need to estimate the relative size of the reflected wave front.

Calling  $\varepsilon^-, \varepsilon^+$  the strengths of the front before and after interaction, to leading order we have

$$\varepsilon^+ = (1 - 2 \tan \theta) \varepsilon^- \quad (4.5)$$

where  $\theta$  is the angle between line segments  $B_1A_2$  and  $A_1A_2$  in Figure 5, and  $s$  is the strength of inner shocks  $A_1C$  or  $A_2C$ .

When the additional front reaches  $B$ , we want its size to be increased by a factor  $\kappa > 1$ . To achieve this goal, we need to perturb the symmetric periodic pattern into an asymmetric periodic pattern as shown in Figure 8.

As in the figure, for simplicity we assume  $h_c = 1, u_c = 0, u_d = r, u_b = s$ . Using the Rankine-Hugoniot condition, we can calculate  $h_b$  and  $h_d$ .

Indeed,  $u_b - u_c = s = \sqrt{-[p(v_b) - p(v_c)](v_b - v_c)}$ .

$$s = \sqrt{-[p'(v_b - v_c) + \frac{1}{2}p''(v_b - v_c)^2 + \frac{1}{6}p^{(3)}(v_b - v_c)^3 + \frac{1}{24}p^{(4)}(v_b - v_c)^4 + \mathcal{O}((v_b - v_c)^5)](v_b - v_c)} \quad (4.6)$$

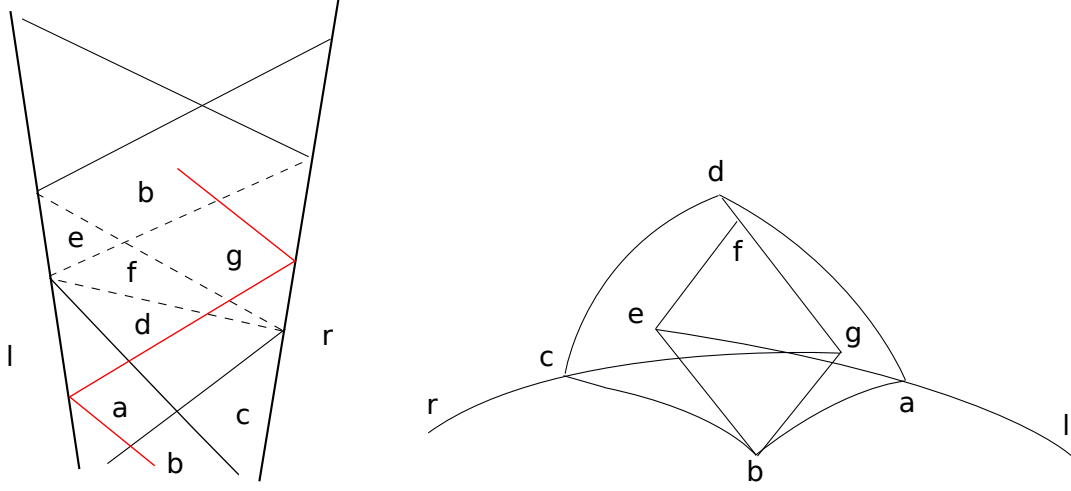


FIGURE 8. Amplification of infinitesimal waves.

By expressing  $v_b - v_c$  in powers of  $s$ , one obtains

$$v_b - v_c = 1 + \sqrt{-p'}s + \frac{p''}{4p'^2}s^2 + \left( \frac{5}{32} \frac{p''^2}{(-p')^{7/2}} + \frac{1}{12} \frac{p^{(3)}}{(-p')^{5/2}} \right) s^3 + \left( -\frac{p''^3}{8p'^5} + \frac{p''p^{(3)}}{8p'^4} - \frac{p^{(4)}}{48p'^3} \right) s^4 + \mathcal{O}(s^5).$$

Considering (1.4), we have

$$h_b - 1 = -s + \frac{1}{96} \frac{p''^2}{(-p')^3} s^3 + \left( \frac{p''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}} \right) s^4 + \mathcal{O}(s^5). \quad (4.7)$$

In a similar way, we obtain

$$h_d - 1 = r - \frac{1}{96} \frac{p''^2}{(-p')^3} r^3 + \left( \frac{p''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}} \right) r^4 + \mathcal{O}(r^5) \quad (4.8)$$

Since  $g$  is the intersection point of two rarefactions, we can calculate the coordinate of  $g$  as

$$\begin{aligned} u_g &= (s + r) - \frac{1}{192} \frac{p''^2}{(-p')^3} (s^3 + r^3) + \mathcal{O}(r^4, s^4), \\ h_g &= 1 + r - s + \frac{1}{192} \frac{p''^2}{(-p')^3} (s^3 - r^3) + \frac{1}{2} \left( \frac{p''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}} \right) (r^4 + s^4) + \mathcal{O}(r^5, s^5). \end{aligned} \quad (4.9)$$

Hence the slope of  $cg$  is

$$\tan \theta = \frac{r - s + \frac{1}{192} \frac{p''^2}{(-p')^3} (s^3 - r^3) + \frac{1}{2} \left( \frac{p''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}} \right) (r^4 + s^4) + \mathcal{O}(s^5, r^5)}{s + r - \frac{29}{192} \frac{p''^2}{(-p')^3} (s^3 + r^3)}. \quad (4.10)$$

For the left boundary shock, we can repeat above process. By assuming  $u_a - u_b = \bar{s}$ ,  $u_a - u_d = \bar{r}$ , we can obtain the slope of  $ae$ , which is similar to (4.10),

$$\tan \theta = \frac{\bar{r} - \bar{s} + \frac{1}{192} \frac{p''^2}{(-p')^3} (\bar{s}^3 - \bar{r}^3) + \frac{1}{2} \left( \frac{p'''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}} \right) (\bar{r}^4 + \bar{s}^4) + \mathcal{O}(\bar{s}^5, \bar{r}^5)}{\bar{s} + \bar{r} - \frac{29}{192} \frac{p''^2}{(-p')^3} (\bar{s}^3 + \bar{r}^3)}. \quad (4.11)$$

Here  $r$  and  $s$  are independent, so we can take different relations between  $r(\bar{r})$  and  $s(\bar{s})$  for the right and left shocks. From the figure, we expect  $r \leq s$ .

For the right boundary shock, we take  $r = s - \left( \frac{p'''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}} \right) s^4$ , To leading order, the slope of the shock curve  $cg$  is

$$\tan \theta = o(s^3).$$

We take  $\bar{r} = \bar{s} - 2J_2\bar{s}^4$ , so the slope of  $ae$  is

$$\tan \theta = (-J_2 + \frac{1}{2}(-p')^{-9/2}J_2)\bar{s}^3 + o(\bar{s}^3).$$

Since  $r - s = \bar{r} - \bar{s}$ , the relation between  $s$  and  $\bar{s}$  is

$$\frac{1}{6(-p')^{9/2}}J_2s^4 = 2J_2\bar{s}^4. \quad (4.12)$$

After one complete set of interaction, the strength of the small wave located at  $B$  is

$$\varepsilon_B = (1 + J_2s^3 + o(s^3))^2 \left( 1 - 2J_2\bar{s}^3 + (-p')^{-9/2}J_2\bar{s}^3 + o(s^3) \right) (1 + o(s^3))\varepsilon_A = (1 + Xs^3 + o(s^3))\varepsilon_A,$$

where

$$X = ((-p')^{-9/2} - 2)J_2 \left( \frac{1}{12(-p')^{9/2}} \right)^{3/4} + 2J_2 > 0.$$

The small wave has been amplified by a factor  $1 + Xs^3 + o(s^3)$ .

By construction, after each period each pair of small compression-rarefaction wavefronts is enlarged by a factor  $\geq \lambda > 1$ . When a pair grows to size  $> \varepsilon$ , we can perform a partial cancellation so that its size remains  $\in [\varepsilon/2, \varepsilon]$ . After this manipulation, we can restrict the specific volume  $v$  to be in the interval  $(v_L, v_U)$ . So the condition (4.1) always holds in the construction.

Since the total number of small wave-fronts is infinite, after several periods a larger and larger number of pairs (compression + rarefaction) reaches size  $> \varepsilon/2$ . Hence, as  $t \rightarrow \infty$ , the total variation of this approximate solution grows without bounds.

## APPENDIX

Some detailed calculations about the slope of shock curves are given below.

As in the figure, for simplicity we assume  $h_c = 1, u_c = 0, u_d = r, u_b = s$ . By R-H condition,  $u_b - u_c = s = \sqrt{-[p(v_b) - p(v_c)](v_b - v_c)}$ . By doing Taylor expansion, we have

$$s = \sqrt{-[p'(v_b - v_c) + \frac{1}{2}p''(v_b - v_c)^2 + \frac{1}{6}p^{(3)}(v_b - v_c)^3 + \frac{1}{24}p^{(4)}(v_b - v_c)^4 + \mathcal{O}((v_b - v_c)^5)](v_b - v_c)} \quad (4.13)$$

We want to express  $v_b - v_c$  in powers of  $s$ . Assume  $v_b - v_c = 1 + As + Bs^2 + Cs^3 +Ds^4 + \mathcal{O}(s^5)$ , then compare the coefficients of  $s^2, s^3, s^4, s^5, s^6$  to determine constants  $A, B, C, D$ .

$$\begin{aligned}
s^2 &= -[p'(As + Bs^2 + Cs^3 +Ds^4 + \mathcal{O}(s^5)) + \frac{1}{2}p''(As + Bs^2 + Cs^3 +Ds^4 + \mathcal{O}(s^5))^2 \\
&+ \frac{1}{6}p^{(3)}(As + Bs^2 + Cs^3 +Ds^4 + \mathcal{O}(s^5))^3 + \frac{1}{24}p^{(4)}(As + Bs^2 + Cs^3 +Ds^4 + \mathcal{O}(s^5))^4 + \mathcal{O}(s^5)] \\
&\cdot (As + Bs^2 + Cs^3 +Ds^4 + \mathcal{O}(s^5))
\end{aligned} \tag{4.14}$$

Coefficient for  $s^2$ :

$$\begin{aligned}
1 &= -A^2p', \\
A &= \pm(-p')^{-1/2}.
\end{aligned}$$

Coefficient for  $s^3$ :

$$\begin{aligned}
0 &= -A(Bp' + \frac{1}{2}A^2p'') - BAp', \\
B &= -\frac{p''}{4p'}A^2 = \frac{p''}{4p'^2}.
\end{aligned}$$

Coefficient for  $s^4$ :

$$\begin{aligned}
0 &= -A(Cp' + p''AB + \frac{1}{6}p^{(3)}A^3) - B(Bp' + \frac{1}{2}A^2p'') - CAp' \\
&= -2ACp' - p''A^2B - \frac{1}{6}p^{(3)}A^4 - B^2p' - \frac{1}{2}A^2Bp''. \\
2ACp' &= \frac{p''^2}{4p'^3} - \frac{p^{(3)}}{6p'^2} - \frac{p''^2}{16p'^3} + \frac{p''^2}{8p'^3} = \frac{5p''^2}{16p'^3} - \frac{p^{(3)}}{6p'^2} \\
C &= \pm \frac{5}{32} \frac{p''^2}{(-p')^{7/2}} \pm \frac{1}{12} \frac{p^{(3)}}{(-p')^{5/2}}.
\end{aligned}$$

Coefficient for  $s^5$ :

$$\begin{aligned}
0 &= -A(Dp' + \frac{1}{2}p''B^2 + p''AC + \frac{1}{6}p^{(3)}3A^2B + \frac{1}{24}p^{(4)}A^4) \\
&\quad -B(Cp' + p''AB + \frac{1}{6}p^{(3)}A^3) \\
&\quad -C(Bp' + \frac{1}{2}A^2p'') \\
&\quad -DAp' \\
2ADp' &= -2BCp' - \frac{3}{2}p''AB^2 - \frac{3}{2}p''A^2C - \frac{2}{3}p^{(3)}A^3B - \frac{1}{24}p^{(4)}A^5, \\
D &= -\frac{BC}{A} - \frac{3p''}{4p'}B^2 - \frac{3p''}{4p'}AC - \frac{1}{3} \frac{p^{(3)}}{p'}A^2B - \frac{1}{48} \frac{p^{(4)}}{p'}A^4 \\
&= -\frac{p''^3}{8p'^5} + \frac{p''p^{(3)}}{8p'^4} - \frac{p^{(4)}}{48p'^3}.
\end{aligned}$$

So by definition of  $h_b$  and values of  $A, B, C, D$ , we have

$$\begin{aligned}
h_b - 1 &= \int_{v_b}^{v_c} \sqrt{-p'} dV = - \int_{v_c}^{v_c + \Delta} \sqrt{-p'} dv \\
&= - \int_{v_c}^{v_c + \Delta} \sqrt{-p'} + \frac{1}{2}(-p')^{-1/2}(-p'')(v - v_c) + \frac{1}{2}[-\frac{1}{4}(-p')^{-3/2}p''^2 - \frac{1}{2}(-p')^{-1/2}p^{(3)}](v - v_c)^2 \\
&\quad + \frac{1}{6}[-\frac{3}{8}(-p')^{-5/2}p''^3 - \frac{1}{2}(-p')^{-3/2}p''p^{(3)} - \frac{1}{4}(-p')^{-3/2}p''p^{(3)} - \frac{1}{2}(-p')^{-1/2}p^{(4)}](v - v_c)^3 dv \\
&= -\sqrt{-p'}\Delta - \frac{1}{2}(-p')^{-1/2}(-p'')\frac{1}{2}\Delta^2 - \frac{1}{6}[-\frac{1}{4}(-p')^{-3/2}p''^2 - \frac{1}{2}(-p')^{-1/2}p^{(3)}]\Delta^3 \\
&\quad - \frac{1}{24}[-\frac{3}{8}(-p')^{-5/2}p''^3 - \frac{3}{4}(-p')^{-3/2}p''p^{(3)} - \frac{1}{2}(-p')^{-1/2}p^{(4)}]\Delta^4 \\
&= -(-p')^{1/2}As + \left(-(-p')^{1/2}B + \frac{1}{4}(-p')^{-1/2}p''A^2\right)s^2 \\
&\quad + \left(-(-p')^{1/2}C + \frac{1}{2}(-p')^{-1/2}p''AB - \frac{1}{6}[-\frac{1}{4}(-p')^{-3/2}p''^2 - \frac{1}{2}(-p')^{-1/2}p^{(3)}](-p')^{-3/2}\right)s^3 \\
&\quad + \left(-(-p')^{1/2}D - \frac{1}{2}(-p')^{-1/2}(-p'')\frac{1}{2}(B^2 + 2AC + \frac{1}{6}[\frac{1}{4}(-p')^{-3/2}p''^2 + \frac{1}{2}(-p')^{-1/2}p^{(3)}]3A^2B \right. \\
&\quad \left. - \frac{1}{24}[-\frac{3}{8}(-p')^{-5/2}p''^3 - \frac{3}{4}(-p')^{-3/2}p''p^{(3)} - \frac{1}{2}(-p')^{-1/2}p^{(4)}]A^4\right)s^4 \\
&= -s + \frac{1}{96}\frac{p''^2}{(-p')^3}s^3 + \left(\frac{p''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}}\right)s^4
\end{aligned} \tag{4.15}$$

Since the rarefaction curves  $dg$  and  $bg$  are perpendicular in the Figure 8, we can solve the following system for  $(u_g, h_g)$ .

$$\begin{cases} x - s = y - (1 - s + \frac{1}{96}\frac{p''^2}{(-p')^3}s^3 + (\frac{p''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}})s^4 + \mathcal{O}(s^5)) \\ x - r = (1 + r - \frac{1}{96}\frac{p''^2}{(-p')^3}r^3 + (\frac{p''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}})r^4 + \mathcal{O}(r^5)) - y \end{cases} \tag{4.16}$$

$$u_g = (s + r) - \frac{1}{192}\frac{p''^2}{(-p')^3}(s^3 + r^3) + \mathcal{O}(r^4, s^4),$$

$$h_g = 1 + r - s + \frac{1}{192}\frac{p''^2}{(-p')^3}(s^3 - r^3) + \frac{1}{2}\left(\frac{p''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}}\right)(r^4 + s^4) + \mathcal{O}(r^5, s^5). \tag{4.17}$$

So the slope of the shock curve  $rg$  at  $g$  is

$$\tan \theta = \frac{r - s + \frac{1}{192}\frac{p''^2}{(-p')^3}(s^3 - r^3) + \frac{1}{2}\left(\frac{p''^3}{64(-p')^{9/2}} + \frac{p''p^{(3)}}{96(-p')^{7/2}}\right)(r^4 + s^4) + \mathcal{O}(s^5, r^5)}{s + r - \frac{29}{192}\frac{p''^2}{(-p')^3}(s^3 + r^3)} \tag{4.18}$$

**Acknowledgments.** The research of the first author was partially supported by NSF, with grant DMS-1411786: “Hyperbolic Conservation Laws and Applications”. The fourth author is supported in part by National Natural Science Foundation of China under grant 11231006, Natural Science Foundation of Shanghai under grant 14ZR1423100 and China Scholarship Council.

## REFERENCES

- [1] P. Baiti and H. K. Jenssen, *Blowup in  $L^\infty$  for a class of genuinely nonlinear hyperbolic systems of conservation laws*, Discrete Contin. Dynam. Systems **7** (2001), no. 4, 837–853.
- [2] A. Bressan, *Hyperbolic systems of conservation laws*, Oxford Lecture Series in Mathematics and its Applications, vol. 20, Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.
- [3] N. S. Bakhvalov, *The existence in the large of a regular solution of a quasilinear hyperbolic system*, Ž. Vyčisl. Mat. i Mat. Fiz. **10** (1970), 969–980 (Russian).
- [4] A. Bressan, G. Chen, and Q. Zhang, *Lack of BV bounds for approximate solutions to the  $p$ -system with large data*, J. Differential Equations **256** (2014), no. 8, 3067–3085.
- [5] A. Bressan, T.P. Liu, and T. Yang,  *$L^1$  stability estimates for  $n \times n$  conservation laws*, Arch. Ration. Mech. Anal. **149** (1999), no. 1, 1–22.
- [6] G. Chen and H. K. Jenssen, *No TVD fields for 1-d isentropic gas flow*, Comm. Partial Differential Equations **38** (2013), no. 4, 629–657.
- [7] C. Dafermos, *Hyperbolic conservation laws in continuum physics*, 3rd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 325, Springer-Verlag, Berlin, 2010.
- [8] R. J. DiPerna, *Existence in the large for quasilinear hyperbolic conservation laws*, Arch. Rational Mech. Anal. **52** (1973), 244–257.
- [9] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math. **18** (1965), 697–715.
- [10] H. K. Jenssen, *Blowup for systems of conservation laws*, SIAM J. Math. Anal. **31** (2000), no. 4, 894–908.
- [11] P. D. Lax, *Hyperbolic systems of conservation laws. II*, Comm. Pure Appl. Math. **10** (1957), 537–566.
- [12] T. Nishida, *Global solution for an initial boundary value problem of a quasilinear hyperbolic system*, Proc. Japan Acad. **44** (1968), 642–646.
- [13] T. Nishida and J. Smoller, *Solutions in the large for some nonlinear hyperbolic conservation laws*, Comm. Pure Appl. Math. **26** (1973), 183–200.
- [14] J. Smoller, *Shock waves and reaction-diffusion equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 258, Springer-Verlag, New York-Berlin, 1983.
- [15] B. Temple, *Systems of conservation laws with invariant submanifolds*, Trans. Amer. Math. Soc. **280** (1983), no. 2, 781–795.

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PA. 16802, USA  
(bressan@math.psu.edu).

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332 USA  
(gchen73@math.gatech.edu).

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PA. 16802, USA  
(zhang.q@math.psu.edu).

DEPARTMENT OF MATHEMATICS, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI 200240, P.R.CHINA  
(zhushengguo@sjtu.edu.cn).