Uniqueness of Conservative Solutions to the Camassa-Holm Equation via Characteristics

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Abstract

The paper provides a direct proof the uniqueness of solutions to the Camassa-Holm equation, based on characteristics. Given a conservative solution $u = u(t, x)$, an equation is introduced which singles out a unique characteristic curve through each initial point. By studying the evolution of the quantities u and $v = 2 \arctan u_x$ along each characteristic, it is proved that the Cauchy problem with general initial data $u_0 \in H^1(I\!R)$ has a unique solution, globally in time.

1 Introduction

The Cauchy problem for the Camassa-Holm equation [5] can be written in the form

$$u_t + (u^2/2)_x + P_x = 0, \tag{1.1}$$

$$u(0, x) = u_0(x). \tag{1.2}$$

The nonlocal source term P is here defined as a convolution:

$$P \doteq \frac{1}{2}e^{-|x|} * \left(u^2 + \frac{u_x^2}{2}\right). \tag{1.3}$$

For any initial data $u_0 \in H^1(I\!R)$, various papers have studied the global existence of solutions [2, 3, 10, 11, 12]. In particular, in [2] a new set of variables was introduced, transforming the equation 1.1 into a semilinear system. This yields a group of conservative (i.e., energy-preserving) solutions $u(t) = S_t u_0$, depending continuously on the initial data w.r.t. suitable norms. In [3] a similar approach was used to construct a semigroup of dissipative solutions. Solutions obtained by this particular transformation of variables are clearly unique. However, in principle, other constructive procedures may yield different solutions to the same Cauchy problem.

Uniqueness is a delicate issue because in general the flow map $(t, u_0) \mapsto S_t u_0$ constructed in [2] is not continuous as a map from $[0, T] \times H^1$ into H^1, neither as a function of time nor of the initial data. Rather, it is continuous from $[0, T] \times H^1$ into spaces with weaker norms such as $\mathbf{L}^2$ or $\mathcal{C}^0$.

The papers [4, 8, 9] have introduced new distances $d^\diamondsuit(\cdot, \cdot)$ on H^1 which render the flow map uniformly Lipschitz continuous on bounded subsets of H^1. These distances satisfy

$$\frac{d}{dt}\Big[d^\diamondsuit(S_t u_0, S_t \tilde{u}_0)\Big] \;\le\; C\, d^\diamondsuit(S_t u_0, S_t \tilde{u}_0), \tag{1.4}$$

for every pair of initial data $u_0, \tilde{u}_0 \in H^1$ and some constant C depending only on the H^1 norm of $u_0, \tilde{u}_0$. We can now consider any map $t \mapsto w(t)$ from an interval $[0, T]$ into H^1 which is Lipschitz continuous w.r.t. the distance $d^\diamondsuit(\cdot, \cdot)$ and satisfies

$$\liminf_{h \to 0+} \frac{1}{h}\, d^\diamondsuit\Big(w(t+h)\,,\; S_h w(t)\Big) \;=\; 0 \qquad \text{for a.e. } t \ge 0. \tag{1.5}$$

An elementary error estimate (see for example [1], p. 26) then yields $w(t) = S_t w(0)$ for every $t \ge 0$. This approach provides some result on the uniqueness of solutions. In practice, however, checking that any conservative solution of (1.1) satisfies the tangency condition (1.5) is not an easy task.

Motivated by a recent paper by Dafermos [6], our present goal is to develop a direct approach to the uniqueness of conservative solutions, based on characteristics. The underlying idea is simply to write a set of ODEs satisfied by u and u_x along the characteristic starting at a given point $\bar{y}$. If this set of equations has a unique solution for a.e. $\bar{y}$, then the entire solution $u(\cdot)$ will be uniquely determined. At this naive level, the approach runs into a fundamental difficulty. Namely, since the solution u is only Hölder continuous, the Cauchy problem

$$\frac{d}{dt}x(t) \;=\; u(t, x(t)), \qquad\qquad x(0) \;=\; \bar{y} \tag{1.6}$$

may well have several solutions (Fig. 1). To overcome this stumbling block, our analysis relies on two key ideas.

- Since we assume that u is a conservative solution, the quantity $w = u_x^2$ provides a measure-valued solution to the balance law

  $$w_t + (uw)_x \;=\; 2(u^2 - P)u_x\,. \tag{1.7}$$

  Because of (1.7), the characteristic curve $t \mapsto x(t)$ satisfies the additional equation

  $$\frac{d}{dt}\int_{-\infty}^{x(t)} u_x^2(t, x)\, dx \;=\; \int_{-\infty}^{x(t)} [2u^2 u_x - 2P u_x](t, x)\, dx\,, \qquad\qquad x(0) = \bar{y}. \tag{1.8}$$

  By itself, (1.8) still does not single out a unique characteristic (think for example of the case where $u(t, x) \equiv 0$). However, combining two equations (1.6) and (1.8) we eventually obtain an integral equation with unique solutions.

- Instead of the variables (t, x), it is convenient to work with an adapted set of variables (t, β), where β is implicitly defined as

  $$x(t, \beta) + \int_{-\infty}^{x(t,\beta)} u_x^2(t, \xi)\, d\xi \;=\; \beta\,. \tag{1.9}$$

In terms of these variables, the solution $u = u(t, \beta)$ becomes globally Lipschitz continuous. Indeed, $|u_\beta| \leq 1$ while $|u_t| \leq C$ for some constant C depending only on the H^1 norm of the solution.

Our analysis eventually shows that, for any conservative solution to the Camassa-Holm equation (1.1), the characteristic curves $t \mapsto x(t, \bar{y})$ as well as the values of u and $v \doteq 2 \arctan u_x$ along these curves can be recovered by a set of integro-differential equations having unique solutions. In turn, this provides a direct proof of the uniqueness of conservative solutions to (1.1), for general initial data $u_0 \in H^1(I\!R)$.

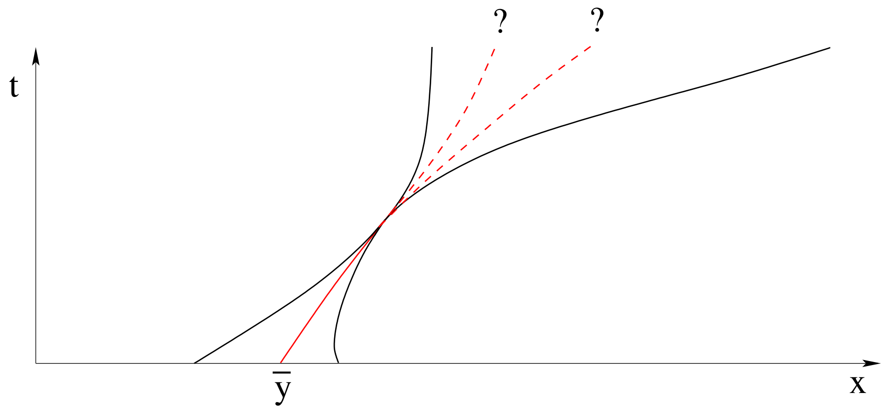

Figure 1: When u is only Hölder continuous, the equation (1.6) does not determine a unique characteristic starting at the point $\bar{y}$.

We emphasize the major difference between the present approach and previous ones. In [2] one starts by constructing a solution to an auxiliary semilinear system. After a suitable variable transformation, this yields a conservative solution to the Camassa-Holm equation (1.1). Here we follow an inverse route. Given a conservative solution $u = u(t, x)$ to (1.1), we introduce a set of auxiliary variables tailored to this particular solution. We then prove that these variables satisfy a particular semilinear system having unique solutions. In turn, this yields the uniqueness of the conservative solution u in the original variables.

The remainder of the paper is organized as follows. In Section 2 we review basic definitions and state our main uniqueness result, Theorem 2. Section 3 establishes the key technical tool (Lemma 2), determining a unique characteristic curve through each initial point. In Section 4 we study how the gradient u_x of a conservative solution varies along a characteristic, and conclude the proof of the main theorem.

2 Basic definitions and results

To make sense of the source term P, at each time t we require that the function $u(t, \cdot)$ lies in the space $H^1(I\!R)$ of absolutely continuous functions $u \in \mathbf{L}^2(I\!R)$ with derivative $u_x \in \mathbf{L}^2(I\!R)$, endowed with the norm

$$\|u\|_{H^1} \doteq \left(\int_{I\!R} \left[u^2(x) + u_x^2(x) \right] dx \right)^{1/2}.$$

For $u \in H^1(I\!R)$, Young's inequality ensures that

$$P \;=\; (1-\partial_x^2)^{-1}\Big(u^2+\frac{u_x^2}{2}\Big) \;\in\; H^1(I\!R).$$

For future use we record the following inequalities, valid for any function $u \in H^1(I\!R)$:

$$\|u\|_{\mathbf{L}^\infty} \;\leq\; \|u\|_{H^1}\,, \tag{2.1}$$

$$\|P\|_{\mathbf{L}^\infty}\,,\quad \|P_x\|_{\mathbf{L}^\infty} \;\leq\; \Big\|\frac{1}{2}e^{-|x|}\Big\|_{\mathbf{L}^\infty}\cdot\Big\|u^2+\frac{u_x^2}{2}\Big\|_{\mathbf{L}^1} \;\leq\; \frac{1}{2}\|u\|_{H^1}\,, \tag{2.2}$$

$$\|P\|_{\mathbf{L}^2}\,,\quad \|P_x\|_{\mathbf{L}^2} \;\leq\; \Big\|\frac{1}{2}e^{-|x|}\Big\|_{\mathbf{L}^2}\cdot\Big\|u^2+\frac{u_x^2}{2}\Big\|_{\mathbf{L}^1} \;\leq\; \frac{1}{\sqrt{2}}\|u\|_{H^1}\,. \tag{2.3}$$

Definition 1. *By a solution of the Cauchy problem (1.1)-(1.2) on $[0,T]$ we mean a Hölder continuous function $u=u(t,x)$ defined on $[0,T]\times I\!R$ with the following properties. At each fixed t we have $u(t,\cdot)\in H^1(I\!R)$. Moreover, the map $t\mapsto u(t,\cdot)$ is Lipschitz continuous from $[0,T]$ into $\mathbf{L}^2(I\!R)$ and satisfies the initial condition (1.2) together with*

$$\frac{d}{dt}u \;=\; -\,uu_x - P_x \tag{2.4}$$

for a.e. t. Here (2.4) is understood as an equality between functions in $\mathbf{L}^2(I\!R)$.

As shown in [2, 3], as soon as the gradient of a solution blows up, uniqueness is lost, in general. To single out a unique solution, some additional conditions are needed.

For smooth solutions, differentiating (1.1) w.r.t. x one obtains

$$u_{xt}+(uu_x)_x \;=\; \Big(u^2+\frac{u_x^2}{2}\Big)-P\,. \tag{2.5}$$

Multiplying (1.1) by u and (2.5) by u_x, we obtain the two conservation laws with source term

$$\Big(\frac{u^2}{2}\Big)_t+\Big(\frac{u^3}{3}+u\,P\Big)_x \;=\; u_x\,P\,, \tag{2.6}$$

$$\Big(\frac{u_x^2}{2}\Big)_t+\Big(\frac{uu_x^2}{2}-\frac{u^3}{3}\Big)_x \;=\; -\,u_x\,P. \tag{2.7}$$

Summing (2.6) and (2.7), and integrating w.r.t. x, we see that for smooth solutions the total energy

$$E(t) \;\doteq\; \int_{I\!R}\Big(u^2(t,x)+u_x^2(t,x)\Big)\,dx \tag{2.8}$$

is constant in time.

Definition 2. *A solution $u=u(t,x)$ is* conservative *if $w=u_x^2$ provides a distributional solution to the balance law (1.7), namely*

$$\int_0^\infty\!\!\int\Big[u_x^2\varphi_t+uu_x^2\varphi_x+2\,(u^2-P)\,u_x\varphi\Big]\,dxdt+\int u_{0,x}^2(x)\varphi(0,x)\,dx \;=\; 0 \tag{2.9}$$

for every test function $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2)$.

The main result proved in [2], on the global existence of conservative solutions can be stated as follows.

Theorem 1. *For any initial data* $u_0 \in H^1(\mathbb{R})$ *the Camassa-Holm equation has a global conservative solution* $u = u(t,x)$. *More precisely, there exists a family of Radon measures* $\{\mu_{(t)}\,,\ t \in \mathbb{R}\}$, *depending continuously on time w.r.t. the topology of weak convergence of measures, such that the following properties hold.*

(i) The function u *provides a solution to the Cauchy problem (1.1)-(1.2) in the sense of Definition 1.*

(ii) There exists a null set $\mathcal{N} \subset \mathbb{R}$ *with* $meas(\mathcal{N}) = 0$ *such that for every* $t \notin \mathcal{N}$ *the measure* $\mu_{(t)}$ *is absolutely continuous and has density* $u_x^2(t,\cdot)$ *w.r.t. Lebesgue measure.*

(iii) The family $\{\mu_{(t)}\,;\ t \in \mathbb{R}\}$ *provides a measure-valued solution* w *to the linear transport equation with source*

$$w_t + (uw)_x \ = \ 2\,(u^2 - P)\,u_x\,. \tag{2.10}$$

At a time $t \in \mathcal{N}$ the measure $\mu_{(t)}$ has a nontrivial singular part. For a conservative solution u which is not smooth, in general we only know that the energy E in (2.8) coincides a.e. with a constant. Namely,

$$E(t) \ = \ E(0) \quad \text{for} \quad t \notin \mathcal{N}, \qquad\qquad E(t) \ < \ E(0) \quad \text{for} \quad t \in \mathcal{N}.$$

The main purpose of this paper is to prove the uniqueness of the above solution.

Theorem 2. *For any initial data* $u_0 \in H^1(\mathbb{R})$, *the Cauchy problem (1.1)-(1.2) has a unique conservative solution.*

3 Preliminary lemmas

Let $u = u(t,x)$ be a solution of the Cauchy problem (1.1)-(1.2) which satisfies the additional balance law (2.9). As mentioned in the Introduction, it is convenient to work with the adapted coordinates (t,β), related to the original coordinates (t,x) by the integral relation (1.9). At times t where the measure $\mu_{(t)}$ is not absolutely continuous w.r.t. Lebesgue measure, we can define $x(t,\beta)$ to be the unique point x such that

$$x(t,\beta) + \mu_{(t)}(\,]-\infty, x[\,) \ \leq \ \beta \ \leq \ x(t,\beta) + \mu_{(t)}(\,]-\infty, x]). \tag{3.1}$$

Notice that (3.1) and (1.9) coincide at every time where $\mu_{(t)}$ is absolutely continuous with density u_x^2 w.r.t. Lebesgue measure. The next lemma, together with Lemma 3, establishes the Lipschitz continuity of x and u as functions of the variables t, β.

Lemma 1. *Let $u = u(t,x)$ be a conservative solution of (1.1). Then, for every $t \geq 0$, the maps $\beta \mapsto x(t,\beta)$ and $\beta \mapsto u(t,\beta) \doteq u(t,x(t,\beta))$ implicitly defined by (3.1) are Lipschitz continuous with constant* 1. *The map $t \mapsto x(t,\beta)$ is also Lipschitz continuous with a constant depending only on $\|u_0\|_{H^1}$.*

Proof. 1. Fix any time $t \geq 0$. The the map

$$x \;\mapsto\; \beta(t,x) \;\doteq\; x + \int_{-\infty}^{x} u_x^2(t,y)\,dy$$

is right continuous and strictly increasing. Hence it has a well defined, continuous, nondecreasing inverse $\beta \mapsto x(t,\beta)$. If $\beta_1 < \beta_2$, then

$$x(t,\beta_2) - x(t,\beta_1) + \mu_{(t)}\Big(\,]x(t,\beta_1)\,,\; x(t,\beta_2)[\,\Big) \;\leq\; \beta_2 - \beta_1\,. \tag{3.2}$$

This implies

$$x(t,\beta_2) - x(t,\beta_1) \;\leq\; \beta_2 - \beta_1\,,$$

showing that the map $\beta \mapsto x(t,\beta)$ is a contraction.

2. To prove the Lipschitz continuity of the map $\beta \mapsto u(t,\beta)$, assume $\beta_1 < \beta_2$. By (3.2) it follows

$$\begin{aligned} \Big|u(t,x(t,\beta_2)) - u(t,x(t,\beta_1))\Big| \;&\leq\; \int_{x(t,\beta_1)}^{x(t,\beta_2)} |u_x|dx \;\leq\; \int_{x(t,\beta_1)}^{x(t,\beta_2)} \frac{1}{2}(1+u_x^2)dx \\ &\leq\; \frac{1}{2}\Big[x(t,\beta_2) - x(t,\beta_1) + \mu_{(t)}\Big(\,]x(t,\beta_1)\,,\; x(t,\beta_2)[\,\Big)\Big] \;\leq\; \frac{1}{2}(\beta_2 - \beta_1)\,. \end{aligned} \tag{3.3}$$

3. Next, we prove the Lipschitz continuity of the map $t \mapsto x(t,\beta)$. Assume $x(\tau,\beta) = y$. We recall that the family of measures $\mu_{(t)}$ satisfies the balance law (2.10), where for each t the drift u and the source term $2(u^2 - P)u_x$ satisfy

$$\|u\|_{\mathbf{L}^\infty(I\!R)} \;\leq\; C_\infty \;\doteq\; \|u\|_{H^1(I\!R)}\,, \tag{3.4}$$

$$\|2(u^2 - P)u_x\|_{\mathbf{L}^1(I\!R)} \;\leq\; 2\Big(\|u\|_{\mathbf{L}^\infty}\|u\|_{\mathbf{L}^2} + \|P\|_{\mathbf{L}^2}\Big)\|u_x\|_{\mathbf{L}^2} \;\leq\; C_S\,, \tag{3.5}$$

for some constant C_S depending only on the H^1 norm of the solution. Therefore, for $t > \tau$ we have

$$\begin{aligned} \mu_{(t)}\Big(\,]-\infty\,,\; y - C_\infty(t-\tau)[\,\Big) \;&\leq\; \mu_{(\tau)}\Big(\,]-\infty\,,\; y[\,\Big) + \int_\tau^t \|2(u^2-P)u_x\|_{\mathbf{L}^1(I\!R)}\,dt \\ &\leq\; \mu_{(\tau)}\Big(\,]-\infty\,,\; y[\,\Big) + C_S(t-\tau)\,. \end{aligned}$$

Defining $y^-(t) \doteq y - (C_\infty + C_S)(t-\tau)$, we obtain

$$\begin{aligned} y^-(t) + \mu_{(t)}\Big(\,]-\infty\,,\; y^-(t)[\,\Big) \;&\leq\; y - (C_\infty + C_S)(t-\tau) + \mu_{(\tau)}\Big(\,]-\infty\,,\; y[\,\Big) + C_S(t-\tau) \\ &\leq\; y + \mu_{(\tau)}\Big(\,]-\infty\,,\; y[\,\Big) \;\leq\; \beta\,. \end{aligned}$$

This implies $x(t,\beta) \;\geq\; y^-(t)$ for all $t > \tau$. An entirely similar argument yields $x(t,\beta) \;\leq\; y^+(t) \doteq y + (C_\infty + C_S)(t-\tau)$, proving the uniform Lipschitz continuity of the map $t \mapsto x(t,\beta)$. □

The next result, which provides the foundation to all our analysis, shows that characteristics can be uniquely determined by an integral equation combining (1.6) with (1.8).

Lemma 2. *Let $u = u(t,x)$ be a conservative solution of the Camassa-Holm equation. Then, for any $\bar y \in I\!R$ there exists a unique Lipschitz continuous map $t \mapsto x(t)$ which satisfies both (1.6) and (1.8). In addition, for any $0 \le \tau \le t$ one has*

$$u(t,x(t)) - u(\tau,x(\tau)) \;=\; -\int_\tau^t P_x(s,x(s))\,ds\,. \tag{3.6}$$

Proof. 1. Using the adapted coordinates (t,β) as in (1.9), we write the characteristic starting at $\bar y$ in the form $t\mapsto x(t) = x(t,\beta(t))$, where $\beta(\cdot)$ is a map to be determined. By summing the two equations (1.6) and (1.8) and integrating w.r.t. time we obtain

$$x(t)+\int_{-\infty}^{x(t)} u_x^2(t,x)\,dx \;=\; \bar y+\int_{-\infty}^{\bar y} u_{0,x}^2(x)\,dx+\int_0^t\left(u(t,x(t))+\int_{-\infty}^{x(t)}[2u^2u_x-2Pu_x](t,x)\,dx\right)dt\,. \tag{3.7}$$

Introducing the function

$$G(t,\beta)\;\doteq\;\int_{-\infty}^{x(t,\beta)}[u_x+2u^2u_x-2u_xP]\,dx \tag{3.8}$$

and the constant

$$\bar\beta \;=\; \bar y+\int_{-\infty}^{\bar y} u_{0,x}^2(x)\,dx\,, \tag{3.9}$$

we can rewrite the equation (3.7) in the form

$$\beta(t)\;=\;\bar\beta+\int_0^t G(s,\beta(s))\,ds\,. \tag{3.10}$$

2. For each fixed $t\ge 0$, since the maps $x\mapsto u(t,x)$ and $x\mapsto P(t,x)$ are both in $H^1(I\!R)$, the function $\beta\mapsto G(t,\beta)$ defined at (3.8) is uniformly bounded and absolutely continuous. Moreover,

$$G_\beta \;=\; [u_x+2u^2u_x-2u_xP]\,x_\beta \;=\; \frac{u_x+2u^2u_x-2u_xP}{1+u_x^2}\;\in\;[-C,\,C] \tag{3.11}$$

for some constant C depending only on the H^1 norm of u. Hence the function G in (3.8) is uniformly Lipschitz continuous w.r.t. β.

3. Thanks to the Lipschitz continuity of the function G, the existence of a unique solution to the integral equation (3.10) can be proved by a standard fixed point argument. Namely, consider the Banach space of all continuous functions $\beta: I\!R_+\mapsto I\!R$ with weighted norm

$$\|\beta\|_* \;\doteq\; \sup_{t\ge 0}\; e^{-2Ct}|\beta(t)|\,.$$

On this space, we claim that the Picard map

$$(\mathcal P\beta)(t)\;\doteq\;\bar\beta+\int_0^t G(\tau,\beta(\tau))\,d\tau$$

is a strict contraction. Indeed, assume $\|\beta - \tilde{\beta}\|_* = \delta > 0$. This implies

$$|\beta(\tau) - \tilde{\beta}(\tau)| \leq \delta e^{2C\tau} \qquad \text{for all } \tau \geq 0.$$

Hence

$$\begin{aligned}\Big|(\mathcal{P}\beta)(t) - (\mathcal{P}\tilde{\beta})(t)\Big| &= \left|\int_0^t \Big(G(\tau,\beta(\tau)) - G(\tau,\tilde{\beta}(\tau))\Big)\,d\tau\right| \leq \int_0^t C\Big|\beta(\tau) - \tilde{\beta}(\tau)\Big|\,d\tau \\ &\leq \int_0^t C\delta e^{2C\tau}\,d\tau \leq \frac{\delta}{2}\,e^{2Ct}.\end{aligned}$$

We thus conclude that $\|\mathcal{P}\beta - \mathcal{P}\tilde{\beta}\|_* \leq \delta/2$.

By the contraction mapping principle, the integral equation (3.10) has a unique solution.

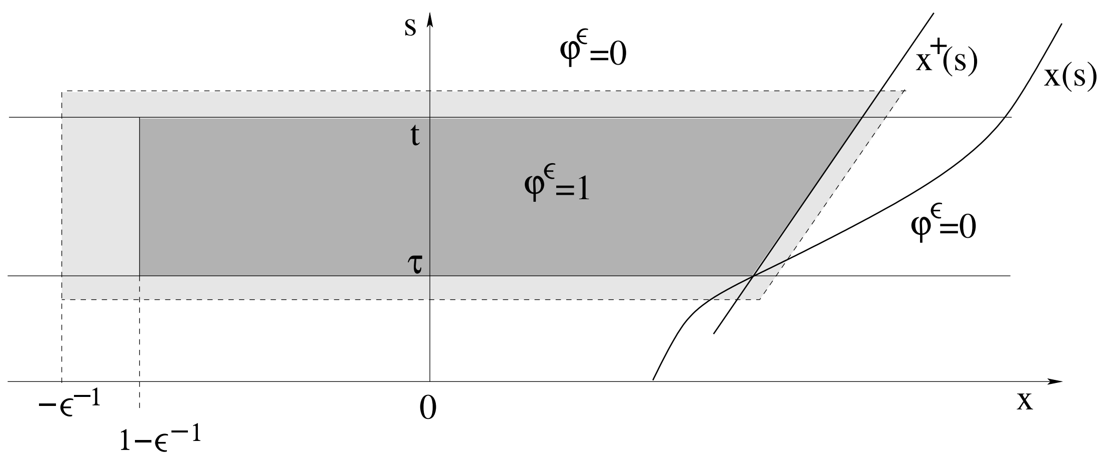

Figure 2: The Lipschitz continuous test function φ^ϵ introduced at (3.15).

4. By the previous construction, the map $t \mapsto x(t) \doteq x(t,\beta(t))$ provides the unique solution to (3.7). Being the composition of two Lipschitz functions, the map $t \mapsto x(t,\beta(t))$ is Lipschitz continuous. To prove that it satisfies the ODE for characteristics (1.6), it suffices to show that (1.6) holds at each time $\tau > 0$ such that

(i) $x(\cdot)$ is differentiable at $t = \tau$, and

(ii) the measure $\mu_{(\tau)}$ is absolutely continuous.

Assume, on the contrary, that $\dot{x}(\tau) \neq u(\tau, x(\tau))$. To fix the ideas, let

$$\dot{x}(\tau) = u(\tau, x(\tau)) + 2\varepsilon_0 \tag{3.12}$$

for some $\varepsilon_0 > 0$. The case $\varepsilon_0 < 0$ is entirely similar. To derive a contradiction we observe that, for all $t \in]\tau, \tau+\delta]$, with $\delta > 0$ small enough one has

$$x^+(t) \doteq x(\tau) + (t-\tau)[u(\tau,x(\tau)) + \varepsilon_0] < x(t)\,. \tag{3.13}$$

We also observe that, since u, P are continuous while $u_x \in \mathbf{L}^2$, by an approximation argument the identity in (2.9) remains valid for any test function $\varphi \in H^1$ with compact support. In particular, this is true if φ is Lipschitz continuous with compact support.

For any $\epsilon > 0$ small, we can thus consider the functions

$$\rho^\epsilon(s,y) \doteq \begin{cases} 0 & \text{if} \quad y \leq -\epsilon^{-1}, \\ (y+\epsilon^{-1}) & \text{if} \quad -\epsilon^{-1} \leq y \leq 1-\epsilon^{-1}, \\ 1 & \text{if} \quad 1-\epsilon^{-1} \leq y \leq x^+(s), \\ 1-\epsilon^{-1}(y-x(s)) & \text{if} \quad x^+(s) \leq y \leq x(s)^+ + \epsilon, \\ 0 & \text{if} \quad y \geq x^+(s)+\epsilon, \end{cases}$$

$$\chi^\epsilon(s) \doteq \begin{cases} 0 & \text{if} \quad s \leq \tau-\epsilon, \\ \epsilon^{-1}(s-\tau+\epsilon) & \text{if} \quad \tau-\epsilon \leq s \leq \tau, \\ 1 & \text{if} \quad \tau \leq s \leq t, \\ 1-\epsilon^{-1}(s-t) & \text{if} \quad t \leq s < t+\epsilon, \\ 0 & \text{if} \quad s \geq t+\epsilon. \end{cases} \tag{3.14}$$

Define

$$\varphi^\epsilon(s,y) \doteq \min\{\varrho^\epsilon(s,y),\, \chi^\epsilon(s)\}. \tag{3.15}$$

Using φ^ϵ as test function in (2.9) we obtain

$$\iint \Big[u_x^2\varphi_t^\epsilon + uu_x^2\varphi_x^\epsilon + 2\,(u^2-P)\,u_x\varphi^\epsilon\Big]\,dxdt = 0. \tag{3.16}$$

We now observe that, if t is sufficiently close to τ, then

$$\lim_{\epsilon\to 0}\int_\tau^t\int_{x^+(s)-\epsilon}^{x^+(s)+\epsilon} u_x^2(\varphi_t^\epsilon + u\varphi_x^\epsilon)\,dxds \geq 0 \tag{3.17}$$

Indeed, for $s \in [\tau+\epsilon,\, t-\epsilon]$ one has

$$0 = \varphi_t^\epsilon + [u(\tau,x(\tau))+\varepsilon_0]\varphi_x^\epsilon \leq \varphi_t^\epsilon + u(s,x)\varphi_x^\epsilon\,,$$

because $u(s,x) < u(\tau,x(\tau))+\varepsilon_0$ and $\varphi_x^\epsilon \leq 0$.

Since the family of measures $\mu_{(t)}$ depends continuously on t in the topology of weak convergence, taking the limit of (3.16) as $\epsilon \to 0$, for $\tau, t \notin \mathcal{N}$ we obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{x(\tau)} u_x^2(\tau,x)\,dx - \int_{-\infty}^{x^+(t)} u_x^2(t,x)\,dx + \int_\tau^t\int_{-\infty}^{x^+(s)}[2u^2u_x - 2u_xP]\,dxds \\ &\qquad + \lim_{\epsilon\to 0}\int_\tau^t\int_{x^+(s)-\epsilon}^{x^+(s)+\epsilon} u_x^2(\varphi_t^\epsilon + u\varphi_x^\epsilon)\,dxds \\ &\geq \int_{-\infty}^{x(\tau)} u_x^2(\tau,x)\,dx - \int_{-\infty}^{x^+(t)} u_x^2(t,x)\,dx + \int_\tau^t\int_{-\infty}^{x^+(s)}[2u^2u_x - 2u_xP]\,dxds \end{aligned} \tag{3.18}$$

In turn, (3.18) implies

$$\begin{aligned} \int_{-\infty}^{x^+(t)} u_x^2(t,x)\,dx &\geq \int_{-\infty}^{x(\tau)} u_x^2(\tau,x)\,dx + \int_\tau^t\int_{-\infty}^{x^+(s)}[2u^2u_x - 2u_xP]\,dxds \\ &= \int_{-\infty}^{x^+(t)} u_x^2(t,x)\,dx + \int_\tau^t\int_{-\infty}^{x(s)}[2u^2u_x - 2u_xP]\,dxds + o_1(t-\tau). \end{aligned}$$

Notice that the last term is a higher order infinitesimal, satisfying $\frac{o_1(t-\tau)}{t-\tau} \to 0$ as $t \to \tau$. Indeed

$$
\begin{aligned}
|o_1(t-\tau)| \;&=\; \left| \int_\tau^t \int_{x^+(s)}^{x(s)} [2u^2 y_x - 2P u_x]\, dxds \right| \;\leq\; \int_\tau^t \int_{x^+(s)}^{x(s)} |2u^2 - 2P|\, |u_x|\, dxds \\
&\leq\; 2\|u^2 - P\|_{\mathbf{L}^\infty} \cdot \int_\tau^t \int_{x^+(s)}^{x(s)} |u_x|\, dxds \\
&\leq\; 2\|u^2 - P\|_{\mathbf{L}^\infty} \int_\tau^t (x(s) - x^+(s))^{1/2}\, \|u_x(s,\cdot)\|_{\mathbf{L}^2}\, ds \;\leq\; C \cdot (t-\tau)^{3/2}.
\end{aligned}
$$

On the other hand, by (3.8) and (3.10) a linear approximation yields

$$
\beta(t) \;=\; \beta(\tau) + (t-\tau) \left[u(\tau, x(\tau)) + \frac{2}{3} u^3(\tau, x(\tau)) - \int_{-\infty}^{x(\tau)} 2u_x P\, dx \right] + o_2(t-\tau)\,, \tag{3.19}
$$

with $\frac{o_2(t-\tau)}{t-\tau} \to 0$ as $t \to \tau$.

For every $t > \tau$ with $t \notin \mathcal{N}$, t sufficiently close to τ, we now have

$$
\begin{aligned}
\beta(t) \;&=\; x(t) + \int_{-\infty}^{x(t)} u_x^2(t,x) dx \\
&>\; x(\tau) + (t-\tau)[u(\tau, x(\tau)) + \varepsilon_0] + \int_{-\infty}^{x^+(t)} u_x^2(t,y) dy \\
&\geq\; x(\tau) + (t-\tau)[u(\tau, x(\tau)) + \varepsilon_0] + \int_{-\infty}^{x(\tau)} u_x^2(\tau,x) dx \\
&\qquad + \int_\tau^t \int_{-\infty}^{x(s)} [2u^2 u_x - 2u_x P]\, dxds + o_1(t-\tau).
\end{aligned} \tag{3.20}
$$

Combining (3.19) and (3.20), we find

$$
\begin{aligned}
&\beta(\tau) + (t-\tau) \left[u(\tau, x(\tau)) + \int_{-\infty}^{x(\tau)} [2u^2 u_x - 2u_x P]\, dx \right] + o_2(t-\tau) \\
&\geq\; \left[x(\tau) + \int_{-\infty}^{x(\tau)} u_x^2(\tau,x)\, dx \right] + (t-\tau)[u(\tau, x(\tau)) + \varepsilon_0] \\
&\qquad + \int_\tau^t \int_{-\infty}^{x(s)} [2u^2 u_x - 2u_x P]\, dxds + o_1(t-\tau).
\end{aligned} \tag{3.21}
$$

Subtracting common terms, dividing both sides by $t - \tau$ and letting $t \to \tau$, we achieve a contradiction. Therefore, (1.6) must hold.

5. We now prove (3.6). By (2.4), for every test function $\phi \in \mathcal{C}_c^\infty(I\!R^2)$ one has

$$
\int_0^\infty \int \left[u\phi_t + \frac{u^2}{2}\phi_x + P_x \phi \right] dxdt + \int u_0(x)\phi(0,x)\, dx \;=\; 0\,. \tag{3.22}
$$

Given any $\varphi \in \mathcal{C}_c^\infty$, let $\phi = \varphi_x$. Since the map $x \mapsto u(t,x)$ is absolutely continuous, we can integrate by parts w.r.t. x and obtain

$$\int_0^\infty \int [u_x\varphi_t + uu_x\varphi_x + P_x\varphi_x]\, dxdt + \int u_{0,x}(x)\varphi(0,x)\, dx \;=\; 0\,. \tag{3.23}$$

By an approximation argument, the identity (3.23) remains valid for any test function φ which is Lipschitz continuous with compact support. For any $\epsilon > 0$ sufficiently small, we thus consider the functions

$$\varrho^\epsilon(s,y) \;\doteq\; \begin{cases} 0 & \text{if} \;\; y \leq -\epsilon^{-1}, \\ y+\epsilon^{-1} & \text{if} \;\; -\epsilon^{-1} \leq y \leq 1-\epsilon^{-1}, \\ 1 & \text{if} \;\; 1-\epsilon^{-1} \leq y \leq x(s), \\ 1-\epsilon^{-1}(y-x(s)) & \text{if} \;\; x(s) \leq y \leq x(s)+\epsilon, \\ 0 & \text{if} \;\; y \geq x(s)+\epsilon, \end{cases}$$

$$\psi^\epsilon(s,y) \;\doteq\; \min\{\varrho^\epsilon(s,y),\, \chi^\epsilon(s)\}, \tag{3.24}$$

where $\chi^\epsilon(s)$ as in (3.14). We now use the test function $\varphi = \psi^\epsilon$ in (3.23) and let $\epsilon \to 0$. Observing that the function P_x is continuous, we obtain

$$\begin{aligned} \int_{-\infty}^{x(t)} u_x(t,x)\, dx \;&=\; \int_{-\infty}^{x(\tau)} u_x(\tau,x)\, dx - \int_\tau^t P_x(s,x(s))\, ds \\ &\quad + \lim_{\epsilon\to 0} \int_{\tau-\epsilon}^{t+\epsilon} \int_{x(s)}^{x(s)+\epsilon} u_x(\psi_t^\epsilon + u\psi_x^\epsilon) dxds\,. \end{aligned} \tag{3.25}$$

To complete the proof it suffices to show that the last term on the right hand side of (3.25) vanishes. Since $u_x \in \mathbf{L}^2$, Cauchy's inequality yields

$$\left| \int_\tau^t \int_{x(s)}^{x(s)+\epsilon} u_x(\psi_t^\epsilon + u\psi_x^\epsilon) dxds \right| \;\leq\; \int_\tau^t \left(\int_{x(s)}^{x(s)+\epsilon} |u_x|^2 dx \right)^{1/2} \left(\int_{x(s)}^{x(s)+\epsilon} (\psi_t^\epsilon + u\psi_x^\epsilon)^2 dx \right)^{1/2} ds\,. \tag{3.26}$$

For each $\epsilon > 0$ consider the function

$$\eta_\epsilon(s) \;\doteq\; \left(\sup_{x\in\mathbb{R}} \int_x^{x+\epsilon} u_x^2(s,y)\, dy \right)^{1/2}. \tag{3.27}$$

Observe that all functions η_ϵ are uniformly bounded. Moreover, as $\epsilon \to 0$ we have $\eta_\epsilon(t) \downarrow 0$ pointwise at a.e. time t. Therefore, by the dominated convergence theorem,

$$\lim_{\epsilon\to 0} \int_\tau^t \left(\int_{x(s)}^{x(s)+\epsilon} u_x^2(s,x) dx \right)^{1/2} ds \;\leq\; \lim_{\epsilon\to 0} \int_\tau^t \eta_\epsilon(s) ds \;=\; 0\,. \tag{3.28}$$

On the other hand, for every time $s \in [\tau, t]$ by construction we have

$$\psi_x^\epsilon(s,y) \;=\; \epsilon^{-1}, \qquad \psi_t^\epsilon(s,y) + u(s,x(s))\psi_x^\epsilon(s,y) \;=\; 0 \qquad \text{for} \;\; x(s) < y < x(s)+\epsilon\,.$$

This implies

$$\int_{x(s)}^{x(s)+\epsilon} |\psi_t^\epsilon(s,y)+u(s,y)\psi_x^\epsilon(s,y)|^2 dy \;=\; \epsilon^{-2}\int_{x(s)}^{x(s)+\epsilon} |u(s,y)-u(s,x(s))|^2 dy$$

$$\leq\; \epsilon^{-1}\cdot\left(\max_{x(s)\leq y\leq x(s)+\epsilon}|u(s,y)-u(s,x(s))|\right)^2 \;\leq\; \epsilon^{-1}\cdot\left(\int_{x(s)}^{x(s)+\epsilon}|u_x(s,y)|\,dy\right)^2 \tag{3.29}$$

$$\leq\; \epsilon^{-1}\cdot\left(\epsilon^{1/2}\cdot\|u_x(s)\|_{\mathbf{L}^2}\right)^2 \;\leq\; \|u(s)\|_{H^1}\,.$$

Together, (3.28) and (3.29) prove that the integral in (3.26) approaches zero as $\epsilon\to 0$. We now estimate the integral near the corners of the domain:

$$\left|\left(\int_{\tau-\epsilon}^{\tau}+\int_t^{t+\epsilon}\right)\int_{x(s)}^{x(s)+\kappa\epsilon} u_x(\psi_t^\epsilon+u\psi_x^\epsilon)dxds\right|$$

$$\leq\; \left(\int_{\tau-\epsilon}^{\tau}+\int_t^{t+\epsilon}\right)\left(\int_{x(s)}^{x(s)+\epsilon}|u_x|^2dx\right)^{1/2}\left(\int_{x(s)}^{x(s)+\epsilon}(\psi_t^\epsilon+u\psi_x^\epsilon)^2dx\right)^{1/2} ds \tag{3.30}$$

$$\leq\; 2\epsilon\cdot\|u(s)\|_{H^1}\cdot\left(\int_{x(s)}^{x(s)+\epsilon}4\epsilon^{-2}\left(1+\|u\|_{\mathbf{L}^\infty}\right)^2\,dx\right)^{1/2} \;\leq\; C\,\epsilon^{1/2}\;\to\;0$$

as $\epsilon\to 0$. The above analysis has shown that

$$\lim_{\epsilon\to 0}\int_{\tau-\epsilon}^{t+\epsilon}\int_{x(s)}^{x(s)+\epsilon} u_x(\psi_t^\epsilon+u\psi_x^\epsilon)dxds \;=\; 0.$$

Therefore from (3.25) we recover (3.6).

6. Finally, we prove uniqueness of $x(t)$. Assume there are different $x_1(t)$ and $x_2(t)$, both satisfying (1.8) together with the characteristic equation (1.6). Choose measurable functions β_1 and β_2 so that $x_1(t)=x(t,\beta_1(t))$ and $x_2(t)=x(t,\beta_2(t))$. Then $\beta_1(\cdot)$ and $\beta_2(\cdot)$ satisfy (3.10) with the same initial condition. This contradicts with the uniqueness of β proved in step **3**. □

Relying on (3.6) we can now show the Lipschitz continuity of u w.r.t. t, in the auxiliary coordinate system.

Lemma 3. *Let $u=u(t,x)$ be a conservative solution of (1.1). Then the map $(t,\beta)\mapsto u(t,\beta)\doteq u(t,x(t,\beta))$ is Lipschitz continuous, with a constant depending only on the norm $\|u_0\|_{H^1}$.*

Proof. Using (3.3), (3.10), and (3.6), and we obtain

$$\left|u(t,x(t,\bar\beta))-u(\tau,\bar\beta)\right| \;\leq\; \left|u(t,x(t,\bar\beta))-u(t,x(t,\beta(t)))\right|+\left|u(t,x(t,\beta(t)))-u(\tau,x(\tau,\beta(\tau)))\right|$$

$$\leq\; \frac{1}{2}|\beta(t)-\bar\beta|+(t-\tau)\|P_x\|_{\mathbf{L}^\infty} \;\leq\; (t-\tau)\cdot(\frac{1}{2}\|G\|_{\mathbf{L}^\infty}+\|P_x\|_{\mathbf{L}^\infty}).$$

□

The next result shows that the solutions $\beta(\cdot)$ of (3.10) depend Lipschitz continuously on the initial data.

Lemma 4. *Let u be a conservative solution to (1.1). Call $t \mapsto \beta(t;\tau,\bar\beta)$ the solution to the integral equation*

$$\beta(t) \ = \ \bar\beta + \int_\tau^t G(\tau,\beta(\tau))\,d\tau, \tag{3.31}$$

with G as in (3.8). Then there exists a constant C such that, for any two initial data $\bar\beta_1,\bar\beta_2$ and any $t,\tau \geq 0$ the corresponding solutions satisfy

$$|\beta(t;\tau,\bar\beta_1) - \beta(t;\tau,\bar\beta_2)| \ \leq \ e^{C|t-\tau|}\,|\bar\beta_1 - \bar\beta_2|. \tag{3.32}$$

Proof. Assume $\tau < t$. By (3.31) it follows

$$\begin{aligned}
|\beta(t;\tau,\bar\beta_1) - \beta(t;\tau,\bar\beta_2)| \ &= \ \left|\bar\beta_1 - \bar\beta_2 + \int_\tau^t G(s,\beta(s;\tau,\bar\beta_1)) - G(s,\beta(s;\tau,\bar\beta_2))ds\right| \\
&\leq \ |\bar\beta_1 - \bar\beta_2| + \int_\tau^t \Big|G(s,\beta(s;\tau,\bar\beta_1)) - G(s,\beta(s;\tau,\bar\beta_2))\Big|\,ds \\
&\leq \ |\bar\beta_1 - \bar\beta_2| + C\int_\tau^t \Big|\beta(s;\tau,\bar\beta_1) - \beta(s;\tau,\bar\beta_2)\Big|\,ds \\
&\leq \ |\bar\beta_1 - \bar\beta_2| e^{C(t-\tau)},
\end{aligned} \tag{3.33}$$

where the last inequality is obtained using Gronwall's lemma. The case $t < \tau$ is entirely similar. □

Lemma 5. *Assume $u \in H^1(I\!R)$ and define the convolution P as in (1.3). Then P_x is absolutely continuous and satisfies*

$$P_{xx} \ = \ P - \left(u^2 + \frac{1}{2}u_x^2\right). \tag{3.34}$$

Proof. The function $\phi(x) = e^{-|x|}/2$ satisfies the distributional identity

$$D_x^2\phi \ = \ \phi - \delta_0\,,$$

where δ_0 denotes a unit Dirac mass at the origin. Therefore, for every function $f \in \mathbf{L}^1(I\!R)$, the convolution satisfies

$$D_x^2(\phi * f) \ = \ \phi * f - f\,.$$

Choosing $f = u^2 + u_x^2/2$ we obtain the result. □

4 Proof of Theorem 2

The proof will be worked out in several steps.

1. By Lemmas 1 and 3, the map $(t,\beta) \mapsto (x,u)(t,\beta)$ is Lipschitz continuous. An entirely similar argument shows that the maps $\beta \mapsto G(t,\beta) \doteq G(t,x(t,\beta))$ and $\beta \mapsto P_x(t,\beta) \doteq P_x(t,x(t,\beta))$ are also Lipschitz continuous. By Rademacher's theorem [7], the partial derivatives x_t, x_β, u_t, u_β, G_β, and $P_{x,\beta}$ exist almost everywhere. Moreover, a.e. point (t,β) is a Lebesgue point for these derivatives. Calling $t \mapsto \beta(t,\bar\beta)$ the unique solution to the integral equation (3.10), by Lemma 4 for a.e. $\bar\beta$ the following holds.

(GC) For a.e. $t > 0$, the point $(t,\beta(t,\bar\beta))$ is a Lebesgue point for the partial derivatives $x_t, x_\beta, u_t, u_\beta, G_\beta, P_{x,\beta}$. Moreover, $x_\beta(t,\beta(t,\bar\beta)) > 0$ for a.e. $t > 0$.

If (GC) holds, we then say that $t \mapsto \beta(t,\bar\beta)$ is a **good characteristic**.

2. We seek an ODE describing how the quantities u_β and x_β vary along a good characteristic. As in Lemma 4, we denote by $t \mapsto \beta(t;\tau,\bar\beta)$ the solution to (3.31). If $\tau, t \notin \mathcal{N}$, assuming that $\beta(\cdot;\tau,\bar\beta)$ is a good characteristic, differentiating (3.31) w.r.t. $\bar\beta$ we find

$$\frac{\partial}{\partial\bar\beta}\beta(t;\tau,\bar\beta) \;=\; 1 + \int_\tau^t G_\beta(s,\beta(s;\tau,\bar\beta)) \cdot \frac{\partial}{\partial\bar\beta}\beta(s;\tau,\bar\beta)\, ds \tag{4.1}$$

Next, differentiating w.r.t. $\bar\beta$ the identity

$$x(t,\beta(t;\tau,\bar\beta)) \;=\; x(\tau,\bar\beta) + \int_\tau^t u(s,\, x(s,\beta(t;\tau,\bar\beta)))\, ds$$

we obtain

$$x_\beta(t,\beta(t;\tau,\bar\beta)) \cdot \frac{\partial}{\partial\bar\beta}\beta(t;\tau,\bar\beta) \;=\; x_\beta(\tau,\bar\beta) + \int_\tau^t u_\beta(s,\beta(s;\tau,\bar\beta)) \cdot \frac{\partial}{\partial\bar\beta}\beta(s;\tau,\bar\beta)\, ds. \tag{4.2}$$

Finally, differentiating w.r.t. $\bar\beta$ the identity (3.6), we obtain

$$u_\beta(t,\beta(t;\tau,\bar\beta)) \cdot \frac{\partial}{\partial\bar\beta}\beta(t;\tau,\bar\beta) \;=\; u_\beta(\tau,\bar\beta) + \int_\tau^t P_{x,\beta}(s,\beta(s;\tau,\bar\beta)) \cdot \frac{\partial}{\partial\bar\beta}\beta(s;\tau,\bar\beta)\, ds\,. \tag{4.3}$$

Combining (4.1)–(4.3), we thus obtain the system of ODEs

$$\left\{\begin{array}{rcl}
\dfrac{d}{dt}\left[\dfrac{\partial}{\partial\bar\beta}\beta(t;\tau,\bar\beta)\right] & = & G_\beta(t,\beta(t;\tau,\bar\beta)) \cdot \dfrac{\partial}{\partial\bar\beta}\beta(t;\tau,\bar\beta),\\[3mm]
\dfrac{d}{dt}\left[x_\beta(t,\beta(t;\tau,\bar\beta)) \cdot \dfrac{\partial}{\partial\bar\beta}\beta(t;\tau,\bar\beta)\right] & = & u_\beta(t,\beta(t;\tau,\bar\beta)) \cdot \dfrac{\partial}{\partial\bar\beta}\beta(t;\tau,\bar\beta),\\[3mm]
\dfrac{d}{dt}\left[u_\beta(t,\beta(t;\tau,\bar\beta)) \cdot \dfrac{\partial}{\partial\bar\beta}\beta(t;\tau,\bar\beta)\right] & = & P_{x,\beta}(t,\beta(t;\tau,\bar\beta)) \cdot \dfrac{\partial}{\partial\bar\beta}\beta(t;\tau,\bar\beta).
\end{array}\right. \tag{4.4}$$

In particular, the quantities within square brackets on the left hand sides of (4.4) are absolutely continuous. From (4.4), using Lemma 5 along a good characteristic we obtain

$$\begin{cases} \dfrac{d}{dt}x_\beta + G_\beta x_\beta &= u_\beta\,, \\[2mm] \dfrac{d}{dt}u_\beta + G_\beta u_\beta &= \left[u^2 + \dfrac{1}{2}u_x^2 - P\right]x_\beta \;=\; \left[u^2 + \dfrac{1}{2}\left(\dfrac{1}{x_\beta} - 1\right) - P\right]x_\beta \\[2mm] &= \left[u^2 - P - \dfrac{1}{2}\right]x_\beta + \dfrac{1}{2}\,. \end{cases} \tag{4.5}$$

3. We now go back to the original (t,x) coordinates and derive an evolution equation for the partial derivative u_x along a "good" characteristic curve.

Fix a point $(\tau,\bar{x})$ with $\tau \notin \mathcal{N}$. Assume that $\bar{x}$ is a Lebesgue point for the map $x \mapsto u_x(\tau,x)$. Let $\bar{\beta}$ be such that $\bar{x} = x(\tau,\bar{\beta})$ and assume that $t \mapsto \beta(t;\tau,\bar{\beta})$ is a *good characteristic*, so that (GC) holds. We observe that

$$u_x^2(\tau,x) \;=\; \frac{1}{x_\beta(\tau,\bar{\beta})} - 1 \;\geq\; 0 \qquad\qquad x_\beta(\tau,\bar{\beta}) \;>\; 0\,.$$

As long as $x_\beta > 0$, along the characteristic through $(\tau,\bar{x})$ the partial derivative u_x can be computed as

$$u_x\Big(t, x(t,\beta(t;\tau,\bar{\beta}))\Big) \;=\; \frac{u_\beta(t,\beta(t;\tau,\bar{\beta}))}{x_\beta(t,\beta(t;\tau,\bar{\beta}))}\,. \tag{4.6}$$

Using the two ODEs (4.2)-(4.3) describing the evolution of u_β and x_β, we conclude that the map $t \mapsto u_x(t,x(t,\beta(t;\tau\bar{\beta})))$ is absolutely continuous (as long as $x_\beta \neq 0$) and satisfies

$$\begin{aligned} \frac{d}{dt}u_x(t,x(t,\beta(t;\tau\bar{\beta}))) \;&=\; \frac{d}{dt}\left(\frac{u_\beta}{x_\beta}\right) \;=\; \frac{x_\beta\left\{[u^2 - P - \frac{1}{2}]x_\beta + \frac{1}{2} - u_\beta G_\beta\right\} - u_\beta\left\{u_\beta - x_\beta G_\beta\right\}}{x_\beta^2} \\ &= u^2 - P - \frac{1}{2} + \frac{1}{2x_\beta} - \frac{u_\beta G_\beta}{x_\beta} - \frac{u_\beta^2}{x_\beta^2} + \frac{u_\beta G_\beta}{x_\beta} \;=\; u^2 - P - \frac{1}{2} + \frac{1}{2x_\beta} - \frac{u_\beta^2}{x_\beta^2}. \end{aligned} \tag{4.7}$$

In turn, as long as $x_\beta > 0$ this implies

$$\begin{aligned} \frac{d}{dt}\arctan u_x(t,x(t,\beta(t;\tau\bar{\beta}))) \;&=\; \frac{1}{1+u_x^2}\cdot\frac{d}{dt}u_x \;=\; \left(u^2 - P - \frac{1}{2} + \frac{1}{2x_\beta} - \frac{u_\beta^2}{x_\beta^2}\right)x_\beta \\ &= \left(u^2 - P - \frac{1}{2}\right)x_\beta + \frac{1}{2} - \frac{u_\beta^2}{x_\beta} \;=\; \left(u^2 - u_x^2 - P - \frac{1}{2}\right)x_\beta + \frac{1}{2} \\ &= \left(u^2 - \frac{1}{x_\beta} - P + \frac{1}{2}\right)x_\beta + \frac{1}{2} \;=\; \left(u^2 - P + \frac{1}{2}\right)x_\beta - \frac{1}{2}\,. \end{aligned} \tag{4.8}$$

4. Consider the function

$$v \;\doteq\; \begin{cases} 2\arctan u_x & \text{if} \quad 0 < x_\beta \leq 1, \\ \pi & \text{if} \quad x_\beta = 0. \end{cases} \tag{4.9}$$

Observe that this implies

$$x_\beta \;=\; \frac{1}{1+u_x^2} \;=\; \cos^2\frac{v}{2}\,, \qquad\qquad \frac{u_x}{1+u_x^2} \;=\; \frac{1}{2}\sin v\,, \qquad\qquad \frac{u_x^2}{1+u_x^2} \;=\; \sin^2\frac{v}{2}\,. \tag{4.10}$$

In the following, v will be regarded as a map taking values in the unit circle $\mathcal{S} \doteq [-\pi,\pi]$ with endpoints identified. We claim that, along each good characteristic, the map $t \mapsto v(t) \doteq v(t, x(t, \beta(t;\tau\bar\beta)))$ is absolutely continuous and satisfies

$$\frac{d}{dt}v(t) \;=\; \big(2u^2-2P+1\big)\cos^2\frac{v}{2} - 1\,. \tag{4.11}$$

Indeed, denote by $x_\beta(t)$, $u_\beta(t)$ and $u_x(t) = u_\beta(t)/x_\beta(t)$ the values of x_β, u_β, and u_x along this particular characteristic. By (GC) we have $x_\beta(t) > 0$ for a.e. $t > 0$.

If τ is any time where $x_\beta(\tau) > 0$, we can find a neighborhood $I = [\tau-\delta, \tau+\delta]$ such that $x_\beta(t) > 0$ on I. By (4.8) and (4.10), $v = 2\arctan(u_\beta/x_\beta)$ is absolutely continuous restricted to I and satisfies (4.11). To prove our claim, it thus remains to show that $t \mapsto v(t)$ is continuous on the null set $\mathcal{N}$ of times where $x_\beta(t) = 0$.

Suppose $x_\beta(t_0) = 0$. From the identity

$$u_x^2(t) \;=\; \frac{1-x_\beta(t)}{x_\beta(t)}\,, \tag{4.12}$$

valid as long as $x_\beta > 0$, it is clear that $u_x^2 \to \infty$ as $t \to t_0$ and $x_\beta(t) \to 0$. This implies $v(t) = 2\arctan u_x(t) \to \pm\pi$. Since in $\mathcal{S}$ we identify the points $\pm\pi$, this establishes the continuity of v for all $t \geq 0$, proving our claim.

5. Let now $u = u(t,x)$ be a conservative solution. As shown by the previous analysis, in terms of the variables t, β the quantities x, u, v satisfy the semilinear system

$$\begin{cases} \dfrac{d}{dt}\beta(t,\bar\beta) &= G(t,\beta(t,\bar\beta)),\\[2mm] \dfrac{d}{dt}x(t,\beta(t,\bar\beta)) &= u(t,\beta(t,\bar\beta)),\\[2mm] \dfrac{d}{dt}u(t,\beta(t,\bar\beta)) &= -\,P_x(t,\beta(t,\bar\beta)),\\[2mm] \dfrac{d}{dt}v(t,\beta(t,\bar\beta)) &= \big(2u^2-2P+1\big)\cos^2\frac{v}{2} - 1\,. \end{cases} \tag{4.13}$$

We recall that P and G were defined at (1.3) and (3.8), respectively. The function P admits a representation in terms of the variable β, namely

$$P(x(\beta)) \;=\; \frac{1}{2}\int_{-\infty}^{\infty} \exp\left\{-\left|\int_\beta^{\beta'}\cos^2\frac{v(s)}{2}\,ds\right|\right\}\cdot\left[u^2(\beta')\cos^2\frac{v(\beta')}{2} + \frac{1}{2}\sin^2\frac{v(\beta')}{2}\right]d\beta', \tag{4.14}$$

$$P_x(x(\beta)) \;=\; \frac{1}{2}\left(\int_\xi^{\infty} - \int_{-\infty}^{\xi}\right)\exp\left\{-\left|\int_\xi^{\beta'}\cos^2\frac{v(s)}{2}\,ds\right|\right\}\left[u^2(\beta')\cos^2\frac{v(\beta')}{2} + \frac{1}{2}\sin^2\frac{v(\beta')}{2}\right]d\beta'. \tag{4.15}$$

For every $\bar\beta \in I\!\!R$ we have the initial condition

$$\left\{\begin{array}{rcl} \beta(0,\bar\beta) & = & \bar\beta, \\ x(0,\bar\beta) & = & x(0,\bar\beta), \\ u(0,\bar\beta) & = & u_0(x(0,\bar\beta)), \\ v(0,\bar\beta) & = & 2\arctan u_{0,x}(x(0,\bar\beta)). \end{array}\right. \tag{4.16}$$

By the Lipschitz continuity of all coefficients, the Cauchy problem (4.13), (4.16) has a unique solution, globally defined for all $t \geq 0$, $x \in I\!\!R$.

6. To complete the proof of uniqueness, consider two conservative solutions $u, \tilde u$ of the Camassa-Holm equation (1.1) with the same initial data $u_0 \in H^1(I\!\!R)$. For a.e. $t \geq 0$ the corresponding Lipschitz continuous maps $\beta \mapsto x(t,\beta)$, $\beta \mapsto \tilde x(t,\beta)$ are strictly increasing. Hence they have continuous inverses, say $x \mapsto \beta^*(t,x)$, $x \mapsto \tilde\beta^*(t,x)$.

By the previous analysis, the map $(t,\beta) \mapsto (x,u,v)(t,\beta)$ is uniquely determined by the initial data u_0. Therefore

$$x(t,\beta) \;=\; \tilde x(t,\beta), \qquad\qquad u(t,\beta) = \tilde u(t,\beta).$$

In turn, for a.e. $t \geq 0$ this implies

$$u(t,x) \;=\; u(t,\beta^*(t,x)) \;=\; \tilde u(t,\tilde\beta^*(t,x)) \;=\; \tilde u(t,x).$$

□

References

[1] A. Bressan, *Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy Problem.* Oxford University Press, 2000.

[2] A. Bressan and A. Constantin, Global conservative solutions to the Camassa-Holm equation, *Arch. Rat. Mech. Anal.* **183** (2007), 215-239.

[3] A. Bressan and A. Constantin, Global dissipative solutions to the Camassa-Holm equation, *Analysis and Applications* **5** (2007), 1–27.

[4] A. Bressan and M. Fonte, An optimal transportation metric for solutions of the Camassa-Holm equation, *Methods and Applications of Analysis*, **12** (2005), 191–220.

[5] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993), 1661–1664.

[6] C. Dafermos, Generalized characteristics and the Hunter-Saxton equation. *J. Hyperbolic Diff. Equat.* **8**, (2011) 159–168.

[7] L. C. Evans, *Partial Differential Equations.* Second edition. American Mathematical Society, Providence, RI, 2010.

[8] K. Grunert, H. Holden, and X. Raynaud, Lipschitz metric for the periodic Camassa-Holm equation. *J. Differential Equations* **250** (2011), 1460–1492.

[9] K. Grunert, H. Holden, and X. Raynaud, Lipschitz metric for the Camassa-Holm equation on the line. *Discrete Contin. Dyn. Syst.* **33** (2013), 2809–2827.

[10] H. Holden and X. Raynaud, Global conservative solutions of the Camassa-Holm equation - a Lagrangian point of view. *Comm. Partial Diff. Equat.* **32** (2007), 1511–1549.

[11] Z. Xin and P. Zhang, On the weak solutions to a shallow water equation, *Comm. Pure Appl. Math.* **53** (2000), 1411–1433.

[12] Z. Xin and P. Zhang, On the uniqueness and large time behavior of the weak solutions to a shallow water equation, *Comm. Partial Diff. Equat.* **27** (2002), 1815–1844.