Large solutions for compressible Euler equations

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Compressible Euler equations

Compressible Euler system is a fundamental model in fluid dynamics, used to describe compressible inviscid flow, such as gases. The system was first written down by Euler in 1757, then studied by many great mathematicians.

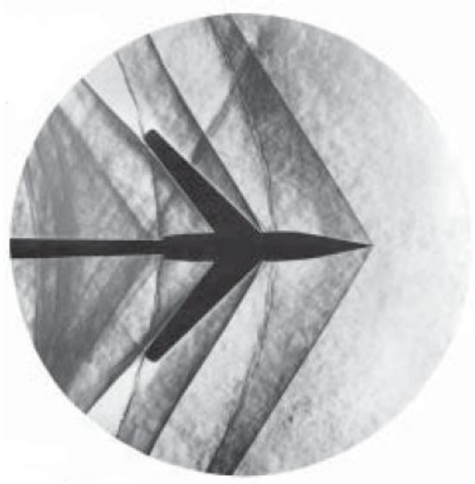

Figure : Shocks near a supersonic body. (Courtesy of NASA)

Compressible Euler system is in the form of hyperbolic conservation laws

1-D hyperbolic conservation laws:

$$\mathbf{u}_t + f(\mathbf{u})_x = 0,$$

with $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.

$$\mathbf{u}(t, x) = (u_1, \cdots, u_n)^T \in \mathbb{R}^n,$$

and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear map.

Initial value problem of hyperbolic conservation laws

Small data 1-d solution for system of conservation laws

- Lax '57: Riemann problems.
- Glimm '65: Global existence for small BV (bounded variation) solutions.
- Other methods: Front tracking scheme, Risebro '92, Bressan '93; Vanishing viscosity, Bianchini-Bressan '05.
- Stability: Bressan-T.P.Liu-T.Yang '99; Uniqueness: Bressan-Goatin-LeFloch '97-'99; Asymptotic behavior, T.P.Liu '78; ...

Major open problems

- Existence for 1-d large solutions for system.
  (This talk is for 1-d large solution of Euler.)
- Existence for multi-d solutions for system.

Note: Small data solution means solution sufficiently close to a constant solution. Large data results have no restriction on the size of solutions.

1-d compressible Euler equations

Eulerian coordinates:

$$\begin{aligned}
\rho_{t'} + (\rho u)_{x'} &= 0\,, && \text{mass}\\
(\rho u)_{t'} + (\rho u^2 + p)_{x'} &= 0\,, && \text{momentum}\\
(\rho u^2/2 + \rho\mathcal{E})_{t'} + \big(\rho u^3/2 + \rho u\mathcal{E} + u\,p\big)_{x'} &= 0\,. && \text{energy}
\end{aligned}$$

Lagrangian coordinates: $x = \int \rho\, dx',\ \ t = t'$.

$$\begin{aligned}
\tau_t - u_x &= 0\,, && \text{mass}\\
u_t + p_x &= 0\,, && \text{momentum}\\
(u^2/2 + \mathcal{E})_t + (u\,p)_x &= 0\,. && \text{energy}
\end{aligned}$$

$$\begin{aligned}
\rho &: \text{density}, & S &: \text{entropy},\\
\tau &: \rho^{-1}\ \text{specific volume}, & \mathcal{E}(\tau,S) &: \text{specific internal energy},\\
u &: \text{fluid velocity}, & p(\tau,S) &: \text{pressure}.
\end{aligned}$$

An equation of state to complete the system. Polytropic ideal gas: $p = k\, e^{\frac{S}{c_V}}\, \tau^{-\gamma}$.

Solutions in two coordinates are equivalent!

Isentropic Euler

Assume entropy S is constant, then mass and momentum equations form a complete system:

$$\begin{cases} \tau_t - u_x &= 0 \\ u_t + p(\tau)_x &= 0 \end{cases}$$

$$p = \tau^{-\gamma}, \quad \gamma > 1\,.$$

A pioneer work of Bernhard Riemann

Bernhard Riemann (1826-1866)

Ueber

die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite.

Von

B. Riemann.

Der Königlichen Societät vorgelegt am 22. November 1859.

Obwohl die Differentialgleichungen, nach welchen sich die Bewegung der Gase bestimmt, längst aufgestellt worden sind, so ist doch ihre Integration fast nur für den Fall ausgeführt worden, wenn die Druckverschiedenheiten als unendlich kleine Bruchtheile des ganzen Drucks betrachtet werden können, und man hat sich bis auf die neuste Zeit begnügt, nur die ersten Potenzen dieser Bruchtheile zu berücksichtigen. Erst ganz vor Kurzem hat Helmholtz auch die Glieder zweiter Ordnung mit in die Rechnung gezogen und daraus die objective Entstehung von Combinationstönen erklärt. Es lassen sich indess für den Fall, dass die anfängliche Bewegung allenthalben in gleicher Richtung stattfindet und in jeder auf diese Richtung senkrechten Ebene Geschwindigkeit und Druck constant sind, die exacten Differentialgleichungen vollständig integriren; und wenn auch zur Erklärung der bisjetzt experimentell festgestellten Erscheinungen die bisherige Behandlung vollkommen ausreicht, so könnten doch, bei den grossen Fortschritten, welche in neuester Zeit durch Helmholtz auch in der experimentellen Behandlung akustischer Fragen gemacht worden sind, die Resultate dieser genaueren Rechnung in nicht allzu ferner Zeit vielleicht der experimentellen Forschung einige Anhaltspunkte gewähren; und dies mag, abgesehen von dem theoretischen Interesse, welches die Behandlung nicht linearer partieller Differentialgleichungen hat, die Mittheilung derselben rechtfertigen.

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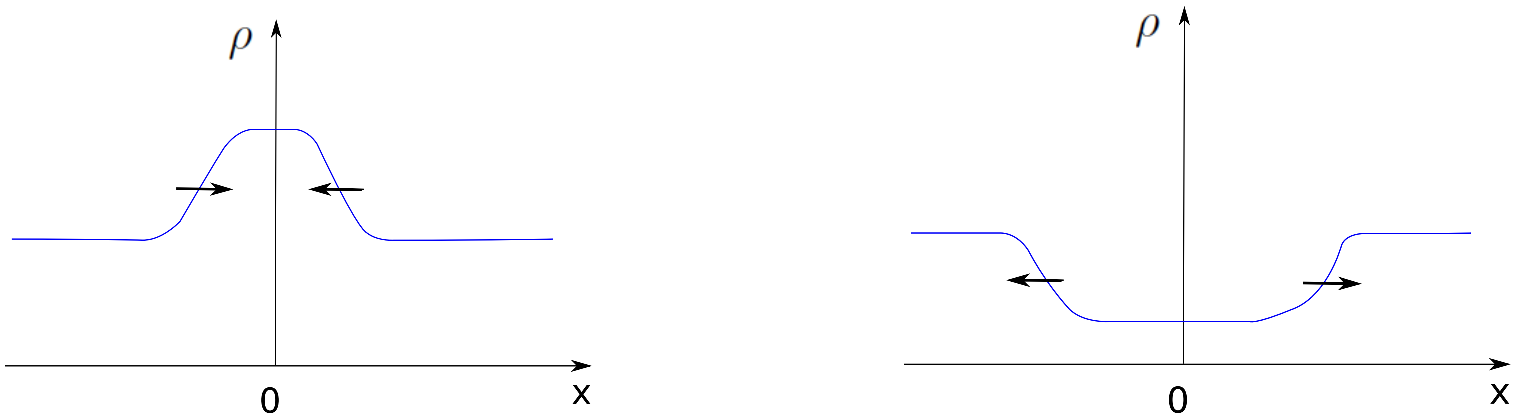
Riemann considered 1-d isentropic gas

- Exact interaction between two centered rarefaction (expansion) waves



For special examples, density ρ approaches zero at the rate

$$\min_x\{\rho(t,x)\} = O\left(\frac{1}{1+t}\right), \qquad \text{Courant-Friedrichs 1948.}$$

- Riemann problem

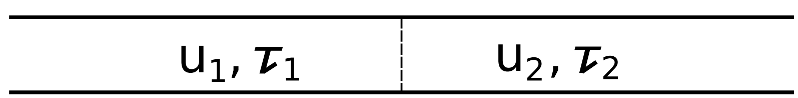


Figure : Gas in a tube

Global-in-time existence for large data solutions of Euler

Large L^∞-existence for isentropic gas: by method of compensated compactness.

DiPerna '83, Ding-Chen-Luo '85-'89, Lions-Perthame-Tadmor-Souganidis '94,'96.

Our target is large BV (bounded variation) existence.

The L^∞ theorem does not provide desired information on structures of solutions. **The desired class of solutions is BV function**, because solutions contain discontinuities/shocks. In this class of solutions, uniqueness is given for small data problem.

Why do we expect large BV existence?

Supported by experiments and numerics; Small BV existence. Large BV results for special cases: Nishida '68, $\gamma = 1$ (isothermal), Nishida-Smoller '73, $\gamma \to 1+$.

Our strategy for 1-d large BV (bounded total variation) existence for isentropic Euler:

1. Find a time-dependent lower bound on density for entropy BV solutions.
2. In a time interval $t \in [0, T]$, assume density is uniformly positive, then find BV estimate for approximate solutions.

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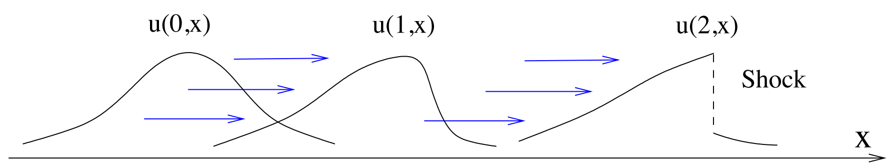
I will show my works in the development of both questions:

i. A sharp lower bound on density for classical solution, which helps constructing **complete singularity (shock) formation results for isentropic and full Euler equations.**
ii. BV-norm estimate for isentropic Euler equations.

1. Singularity formation and lower bound on density

Burgers' equation



$$u_t + u\, u_x = 0, \qquad u(0,x) = u_0(x).$$

- Differentiate it on x,

$$u_{xt} + uu_{xx} + u_x^2 = 0 \overset{v=u_x}{\Longrightarrow} v_t + uv_x + v^2 = 0.$$

Denote $\, ' = \partial_t + u\partial_x$,

$$v' = -v^2 \;\Rightarrow\; v = \frac{v_0}{1 + v_0 t}.$$

- $v_0 < 0 \iff v \to -\infty$ in finite time.

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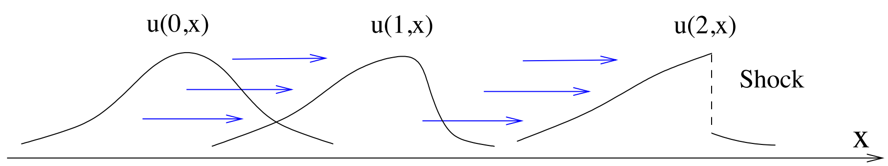

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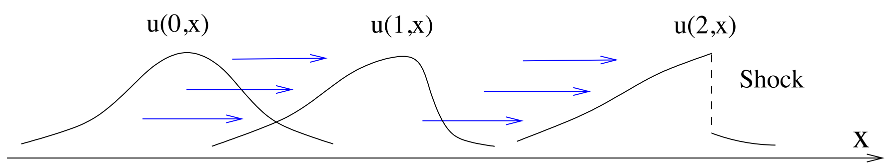

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- $v_0 < 0 \Longleftrightarrow v \to -\infty$ in finite time.
- Compression $\Longleftrightarrow u_x < 0$; Expansion (rarefaction) $\Longleftrightarrow u_x > 0$.
- initial data have compression somewhere $\Longleftrightarrow$ shock forms in finite time.

System with two unknowns

Small data problems: Solved!

Theorem (Lax '64)

For small data solution in strictly hyperbolic system with 2 unknowns, singularity forms in finite time if initial data include compression.

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Open problem:

For (large) solution of Euler equation, can one prove that singularity forms in finite time if and only if initial data include compression somewhere?

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Solved after GC-Pan-Zhu '14.

Isentropic Euler

$$\begin{cases} \tau_t - u_x &= 0, \\ u_t + p(\tau)_x &= 0. \end{cases}$$

$$p = \tau^{-\gamma} \text{ and } \tau = 1/\rho.$$

- Equivalent to

$$w_t + c\, w_x = 0, \qquad z_t - c\, z_x = 0$$

on Riemann invariants

$$w = u + h, \qquad z = u - h,$$

with $h = k_1\, \tau^{-\frac{\gamma-1}{2}}$ and wave speed $c = k_2\, \tau^{-\frac{\gamma+1}{2}}$.

- Apply Lax '64 to isentropic Euler: For $y = \sqrt{c}\, w_x$ & $q = \sqrt{c}\, z_x$,

$$y' = -a(\rho) y^2 \implies y(T) = \frac{y_0}{1 + y_0 \int_0^T a\Big(\rho\big(t, x(t)\big)\Big)\, dt}.$$

$x(t)$ is a forward characteristic, $a(\rho) = K\rho^{\frac{3-\gamma}{4}}$, $' = \partial_t + c\partial_x$.
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To prove singularity formation: One needs to show

$$\int_0^\infty a(\rho)\, dt = \infty. \tag{1}$$

- If (1) is correct, when $y_0 < 0 \implies y$ breaks down.
- For small data solutions, (1) is trivial, when initial density is away from zero.
- However, (1) is **nontrivial** for large data solutions.

Singularity formation for large solution: Now the open problem changes to

$$\text{Is} \quad \int_0^\infty a(\rho)\,dt = \infty, \quad \text{where} \quad a(\rho) = K\rho^{\frac{3-\gamma}{4}} \text{ ?} \tag{2}$$

- Riemann invariants are constant along characteristics, we get $\rho <$ Constant. Lax's theorem directly applies to $\gamma \geq 3$.

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- If $1 < \gamma < 3$, proof of (2) is challenging because ρ might approach zero as $t \to \infty$! (Example: interaction of two strong rarefactions.)
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This problem is resolved in GC-Pan-Zhu '14!

Theorem (Lax '64 & GC-Pan-Zhu '14.)

For isentropic Euler, if C^1-norm of initial data (u_0, ρ_0) is bounded & ρ_0 is uniformly positive, global classical solution exists if & only if

$$\min_{x\in\mathbb{R}} \Big\{ w_x(0,x), z_x(0,x) \Big\} \geq 0\,.$$

From physical point of view, this theorem tells that classical solution breaks down if & only if initial data are forward or backward compressive somewhere.

Existence part was classical result (Longwei Lin '87).

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Optimal bound: In GC '15, I improve lower bound of density to the optimal order $O(\frac{1}{1+t})$, and show $u_x(t,x) < K$ before blowup for some constant K only depending on initial data, although $u_x \to -\infty$ at singularity.

A short proof in GC-Pan-Zhu '14: We first use equations of Riemann invariants

$$w = u + h(\rho), \quad z = u - h(\rho),$$

to estimate maximum rarefaction (upper bounds on y and q). Note density decreases after crossing rarefaction and increases after crossing compression.

$$y' = -a(\rho)y^2\,, \qquad q^{\backprime} = -a(\rho)q^2\,,$$

$$y = \sqrt{c}\,w_x\,, \qquad q = \sqrt{c}\,z_x\,.$$

Lemma (Bounds on maximum rarefaction)

$$y(t,x) \le \max_x\{0, y(0,x)\} \quad \textit{and} \quad q(t,x) \le \max_x\{0, q(0,x)\}.$$

By the conservation of mass $\tau_t = u_x$,

$$2\sqrt{c}\,\tau_t = 2\sqrt{c}\,u_x = (y+q) \le K_1\,.$$

Then when $1 < \gamma < 3$,

$$a(\rho) \ge O(1+t)^{-1} \quad \Rightarrow \quad \int_0^\infty a(\rho)\,dt = \infty.$$

Full Euler equations:

- Decomposition, F.John '74, Yang-Lin-Liu '97, GC '11, GC-Young '12.
- $\gamma \geq 3$, GC-Young-Zhang '13, ($\rho <$ Constant).
- $1 < \gamma < 3$, GC-Pan-Zhu '14, (time-dependent lower bound on ρ).

Theorem (GC-Young-Zhang '13 & GC-Pan-Zhu '14)

For full Euler, when initial data (u_0, τ_0, S_0) are C^1, τ_0 is uniformly positive and entropy S_0 is BV, singularity forms in finite time if

$$\min_x \left\{ y(0,x),\ q(0,x) \right\} < -N.$$

Here N is a positive constant. And for any $\varepsilon > 0$, we find stationary classical solution with

$$\min_x \left\{ y(0,x),\ q(0,x) \right\} = -N + \varepsilon.$$

Compatible to isentropic Euler: $N \to 0$ as $\max\left\{ |S_0'(x)|,\ |S_0''(x)| \right\} \to 0.$

2. BV (bounded variation) norm estimate for large solutions

Large BV existence for isentropic Euler

A review on key steps in small BV theory: Glimm scheme, front tracking scheme ...

- Pairwise wave interactions: Resolved for full Euler with large solutions. Chang-Hsiao 1979, GC-Endres-Jenssen 2012.

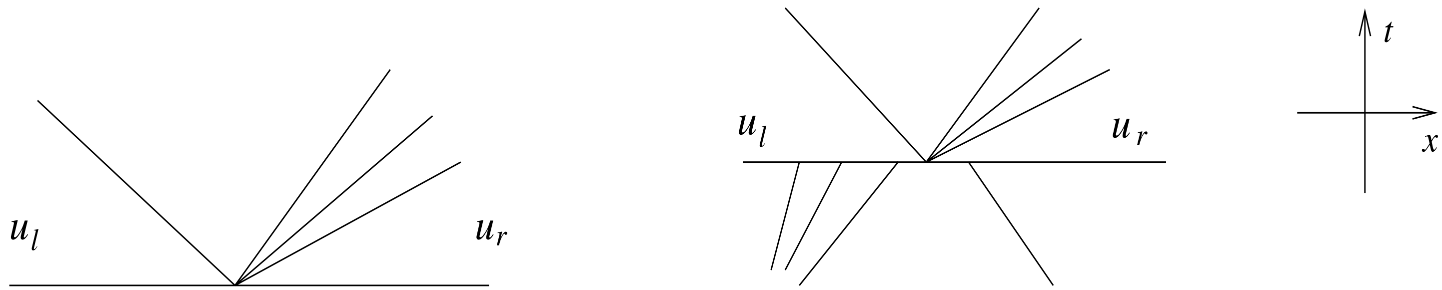

Figure : (Left) Riemann problem; (Right) Pairwise wave interactions

- **(Key step)** Total variation bound on approximation solutions:

$$\textit{Total variation} + C \cdot \textit{Interaction potential} \quad \text{decays}$$

Large BV existence: How to get a BV bound?

- For isothermal gas ($\gamma = 1$), large BV existence is given by T.Nishida '68.

  Total variation bound was found by observing a scalar (TVD) function $\log \rho$*, whose spatial Total Variation Diminishes after pairwise interactions.*

- TVD functions exist in all large BV existence results: Temple system...

- Question 1: Is there a TVD function for isentropic gas ($\gamma > 1$)?

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- Question 1: Is there a TVD function for isentropic gas ($\gamma > 1$)?

We gave the following sad answer

Theorem (GC-H.K.Jenssen, '13)

Isentropic gas with $\gamma > 1$ *doesn't admit non-constant* C^2*-smooth scalar TVD function.*

Question 2: Is that possible to find a BV bound for large solutions by the existing frameworks? Such as front tracking scheme...

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Answer is negative:

Theorem (Bressan-GC-Zhang '14)

For any non-constant C^1 function $\phi(u,\tau)$ and $T>0$, we find a front tracking approximate solution,

$$TV_{t=T}\,\phi = +\infty \qquad \textit{while} \qquad 0 < M_1 < TV_{t=0}\,\phi < M_2,$$

with density uniformly away from vacuum.

Other BV norm instability:

- T.-P. Liu, J. Smoller '80: when solution is close to vacuum.
- C. Tsikkou '09: p-system for elasticity with piecewise linear pressure.

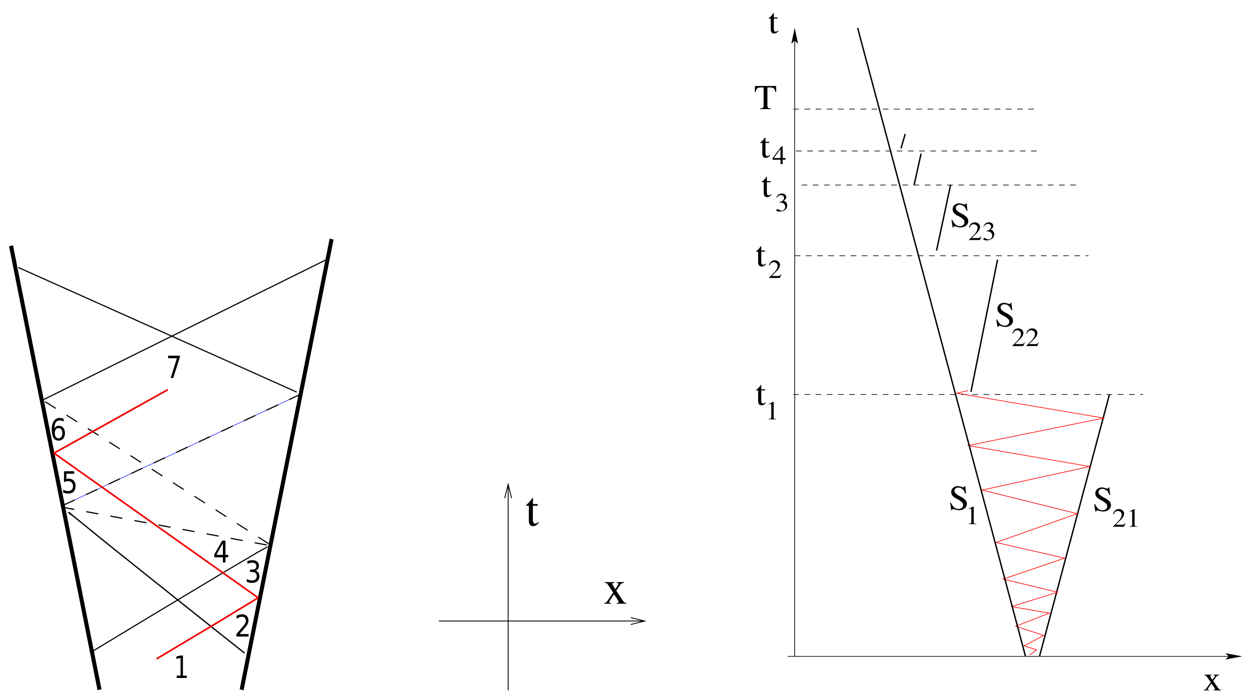


Figure : Black waves form a periodic pattern: Shocks (solid) and rarefactions (dash). "red" waves in the left picture are a large number of infinitesimal waves.

- Wave interactions are exact. Errors are allowed in wave speeds.
- Strength of red waves increases by a constant rate after a cycle (1∼7) .
- Infinite cycles happen in finite time ⇒ Blowup can happen in finite time.
- The density has uniformly positive lower bound.

Periodic pattern : Riemann invariants are $u \pm h(\rho)$.

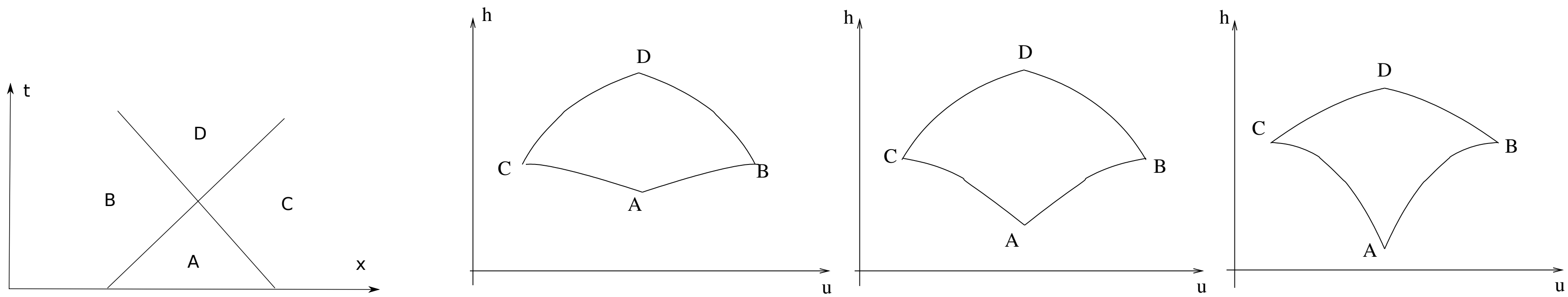
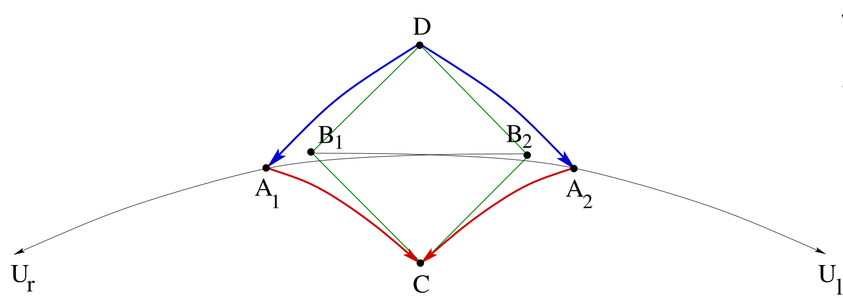
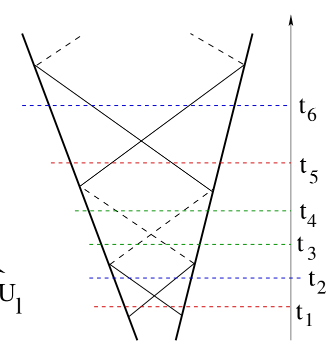


Figure : Interaction of two shocks. Left-most figure is in (x, t) plane. Other figures show three different cases, in the (u, h)-plane. From left to right: $\gamma > 1$; $\gamma = 1$; $0 < \gamma < 1$.





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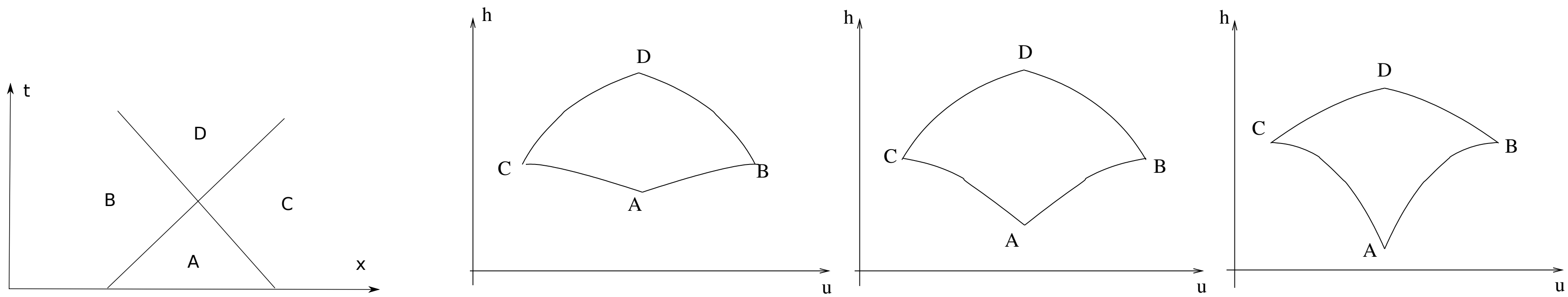

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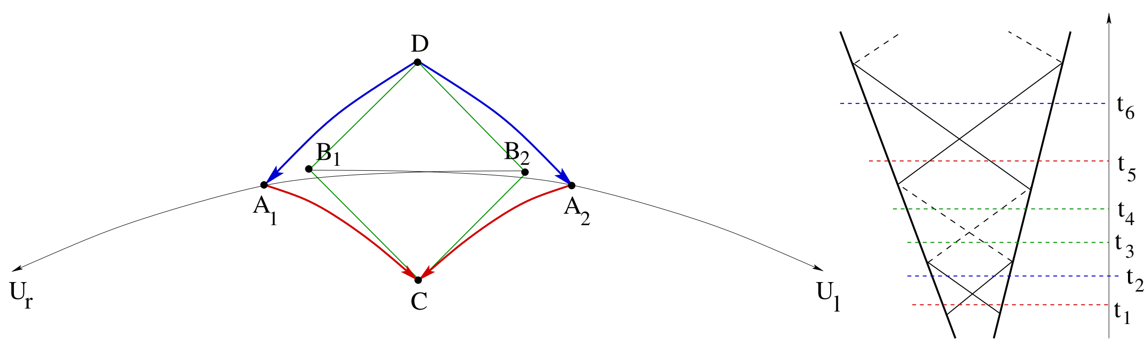

General pressure $p = p(\tau)$, $p_\tau < 0$, $p_{\tau\tau} > 0$

$$\text{Bakhvalov's condition:} \quad 3p_{\tau\tau}^2 \leq 2p_\tau p_{\tau\tau\tau} \quad \text{for all} \quad \tau > 0$$

is satisfied $\Rightarrow$ global BV existence; otherwise our (BV blowup) example exists.

Conclusions

- We completely resolved the singularity formation for isentropic and non-isentropic Euler equations, by extending the theory of Lax. One of the main steps is to find a sharp lower bound on density for classical solution. The next step is to find a lower bound on density for BV solutions.

- We showed that front tracking scheme fail to give useful estimate to large solutions for general Cauchy problem for isentropic Euler equations, even when density is away from vacuum.

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Thank You!