EXISTENCE AND REGULARITY OF SOLUTIONS IN NONLINEAR WAVE EQUATIONS

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ABSTRACT. In this paper, we study the global existence and regularity of Hölder continuous solutions for a series of nonlinear partial differential equations describing nonlinear waves.

CONTENTS

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1. INTRODUCTION

In this paper, we consider the existence and regularity of weak solutions for two families of nonlinear wave equations parameterized by λ

$$u_{tx} + f'(u)\, u_{xx} + \lambda\, f''(u)\, (u_x)^2 = 0\,, \tag{1.1}$$

and

$$u_{tt} - c^2(u)\, u_{xx} - 2\lambda\, c(u)\, c'(u)\, (u_x)^2 = 0\,, \tag{1.2}$$

with constant parameter

$$0 \le \lambda \le 1\,.$$

Here $x \in \mathbb{R}$ is the spatial variable and $t \in \mathbb{R}^+$ is the time variable. The wave speed $c(u) > 0$. Such equations can be formally written as

$$u_{tx} + (f'(u))^{1-\lambda} \left[(f'(u))^{\lambda}\, u_x\right]_x = 0\,, \tag{1.3}$$

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$$u_{tt} - c^{2-2\lambda}(u)\left[c^{2\lambda}(u)u_x\right]_x = 0\,, \tag{1.4}$$

which include several important and interesting models when λ takes different values.

- For equation (1.1) with

$\lambda = 1$: Scalar hyperbolic conservation law.

$\lambda = \frac{1}{2}$: An equation considered in [5]. When $f(u) = \frac{1}{2}u^2$, equation (1.1) is Hunter-Saxton equation modeling nematic liquid crystal [1, 3, 13, 16, 17, 18].

$\lambda = 0$: A wave equation in unitary direction, $u_{tx} + f'(u)u_{xx} = 0$.

- For equation (1.2) with

$\lambda = 1$: Wave equation modeling elasticity

$$u_{tt} - \big(F(u)\big)_{xx} = 0, \quad \text{with} \quad F(u) = \int c^2(u)du\,,$$

or isentropic Euler equations in Lagrangian coordinates, also called p-system:

$$\begin{aligned} u_t - \omega_x &= 0 \\ \omega_t - F_x &= 0 \end{aligned} \tag{1.5}$$

with $\omega = \int u_t\,dx$, and F denotes pressure. See [12] for details.

$\lambda = \frac{1}{2}$: Variational wave equation modeling nematic liquid crystal [4, 13, 14, 23].

$$u_{tt} - c(u)(c(u)u_x)_x = 0\,. \tag{1.6}$$

$\lambda = 0$: A nonlinear wave equation

$$u_{tt} - c^2(u)u_{xx} = 0$$

which is the one dimensional case of

$$u_{tt} - c^2(u)\Delta u = 0\,, \tag{1.7}$$

which was studied in [21].

One common feature of these systems is the finite time gradient blowup of solutions even with smooth initial data, when $0 < \lambda \le 1$, [7, 14, 16, 19]. The motivation why we connect all these equations together is to understand the variation of regularity for weak solutions of these wave equations as λ changes. The first appearance of Equation (1.2) was in [15].

We notice that the regularities of weak solutions for equations in the form of (1.1) (or (1.2)) with $\lambda = 0$, $\frac{1}{2}$ and 1, respectively, are totally different. We summarize the existing results on regularity of weak solutions for these three cases in the following table.

- When $\lambda = 1$, (1.1) and (1.2) can be written in the form of hyperbolic conservation laws. It is well known that solutions in these equations in general have discontinuities (shock waves) even when initial data are smooth, c.f. [6, 7, 8, 9, 12, 19]. BV existence for solution of (1.5) with small amplitude is available in [20].
- When $\lambda = \frac{1}{2}$, the Hunter-Saxton equation and variational wave equation both only have Hölder continuous solutions with exponent $1/2$ because of the possible gradient blowup, c.f. [4, 5, 17] for global existence and [11, 14, 16] for gradient blowup.
- When $\lambda = 0$, there is still no global existence for classical solutions available for general large initial data. But we tend to expect that the solution in this case has better regularity than solutions in previous two cases, because of the global-in-time existence for classical radially symmetric small solution in [21] and the study in this paper for large data solution.

By the discussion for three cases with $\lambda = 0$, $\frac{1}{2}$ and 1, it is very tentative for us to guess that the solution for (1.1) or (1.2) is more regular when λ is decreasing. More intuitively, as λ decreases, i.e. "more" $f'(u)$ or $c(u)$ comes out of the bracket in (1.3) or (1.4), we conject that weak solution of (1.1) or (1.2) has better regularity.

In this paper, we partially prove this conjecture. First, we show that the conjecture is true for (1.1) with $\lambda \in (0, \frac{1}{2}]$ by constructing weak solutions whose regularities vary on λ. Especially, when $\lambda \in (0, \frac{1}{3}] \cup \frac{1}{2}$, the solution is Hölder continuous on both x and t with exponent $1 - \lambda$.

Secondly, we provide some numeric evidences supporting that the solution for wave equation (1.2) with $\lambda \in (0, \frac{1}{3}]$ is Hölder continuous with exponent $1 - \lambda$ when gradient blowup happens. Especially, loosely speaking, when λ is very close to zero, Hölder space with exponent $1 - \lambda$ is getting "close" to the space of C^1 functions.

In fact, for (1.2), we construct a semi-linear system, then the problem whether solution $u(x, t)$ is Hölder continuous with exponent $1 - \lambda$ is changed to an equivalent problem whether variables p and q defined in (3.9) in the semi-linear system are bounded away from zero and infinity, which can be more easily tested by numerical methods than the first problem. We do numerical experiments on several examples, in all of which p and q are indeed bounded away from zero and infinity, although gradient blowup happens in finite time.

We expect this work can help unveiling the mystery in Hölder continuous solutions for quasi-linear hyperbolic systems. To our limit of knowledge, studies on Hölder continuous solutions in wave equations are still very limited. Especially the classification of equations whose solutions are Hölder continuous with different exponents is wide open.

In next two subsections, we introduce the main results for (1.1) and (1.2).

1.1. **Global existence for (1.1) with $\lambda \in (0, \frac{1}{2}]$.** For (1.1), we focus on the case when $\lambda \in (0, \frac{1}{2}]$. The energy law for (1.1) for the smooth solution is

$$\left(|u_x|^{\frac{1}{\lambda}}\right)_t + \left(f'(u)|u_x|^{\frac{1}{\lambda}}\right)_x = 0\,. \tag{1.8}$$

In this paper, we consider the initial boundary value problems for (1.1) on the region $(x, t) \in \mathbb{R}^+ \times \mathbb{R}^+$ when $\lambda \in (0, \frac{1}{2}]$, with initial and boundary conditions

$$u(0, t) = 0, \qquad u(x, 0) =: u_0(x) \in W^{1,\frac{1}{\lambda}}_{loc}(\mathbb{R}^+)\,, \tag{1.9}$$

and a compatibility condition

$$u_0(0) = 0 \quad \text{and} \quad u_0'(0) = 0\,. \tag{1.10}$$

Here $W^{1,\frac{1}{\lambda}}_{loc}(\mathbb{R}^+)$ is the Sobolev space with standard notation.

Throughout this paper, we assume that $f(u)$ is a C^2 function satisfying

$$f'(0) \geq 0, \qquad |f''(u_1) - f''(u_2)| \leq L|u_1 - u_2|\,, \qquad \forall\, u_1,\, u_2 \in \mathbb{R} \tag{1.11}$$

for some constant L, and $|f''(u)|$ is uniformly bounded above. The assumption that $f'(0) \geq 0$ protects that the wave on the boundary $x = 0$ does not flow in an outward direction.

We first define the weak solution.

Definition 1.1. (Weak solution) *The function $u(x, t)$, defined for all $(x, t) \in \mathbb{R}^+ \times \mathbb{R}^+$, is a weak solution for (1.1), if initial and boundary conditions (1.9) and (1.10) are satisfied pointwisely and*

    i. *The equation (1.1) is satisfied in the weak sense*

$$\int_0^\infty \int_0^\infty \left\{ -u_x\left(\phi_t + f'(u)\,\phi_x\right) + (\lambda - 1)\,f''(u)u_x^2\,\phi \right\} dx\,dt = 0, \tag{1.12}$$

    for any test function $\phi \in C_c^1(\mathbb{R}^+ \times \mathbb{R}^+)$.

    ii. *For any fixed $t > 0$, the function $u(\cdot, t)$ is in $W^{1,\frac{1}{\lambda}}_{loc}(\mathbb{R}^+)$, hence is locally Hölder continuous with exponent $1 - \lambda$ by the Sobolev embedding Theorem.*

Then we give the main theorem in this paper for (1.1).

Theorem 1. *The initial boundary value problem (1.1)(1.9)(1.10) with $\lambda \in (0, \frac{1}{2}]$ exists a weak solution $u(x, t)$ under Definition 1.1. When $\lambda \in (0, \frac{1}{3}] \cup \frac{1}{2}$, $u(x, t)$ is locally Hölder continuous on both x and t with exponent $1 - \lambda$.*

To prove Theorem 1, inspired by the energy dependent characteristic coordinates introduced in [2] for Camassa-Holm equation and in [4] for variational wave equation, we introduce an independent variable Y, which dilates the possible gradient blowup due to the concentration of characteristics. Then we establish a semi-linear system for some unknowns on independent variables Y and time. One crucial unknown is

$$\xi = \frac{(1 + u_x^2)^{\frac{1}{2\lambda}}}{Y_x}\,,$$

where Y_x measures the dilation rate of characteristics. By showing ξ is bounded, which can be roughly understood as that the dilation rate of characteristics Y_x is balanced by the energy density, we can prove the global existence for the semi-linear system when $\lambda \in (0, \frac{1}{2}]$. Finally, using an inverse transformation, we construct the weak solution for (1.1). When $\lambda \in (0, \frac{1}{3}] \cup \frac{1}{2}$, the solution has better regularity.

1.2. **Analysis on nonlinear wave equation** (1.2). When $\lambda = \frac{1}{2}$, for (1.2), energy conservative Hölder continuous solutions with exponent $1/2$ has been proved in [4] by introducing new characteristic variables as independent variables. This method has also been used for a series of variational wave equations in nematic liquid crystal [10, 11, 24, 25]. In these results and also our paper, the wave speed $c(u)$ is assumed to be uniformly positive and bounded.

In this paper, inspired by (1.1), we derive a semi-linear system for (1.2). Although we still cannot conclude any global existence results, we expect this system could create a framework in proving the global existence of Hölder continuous solutions for (1.2) in the future.

When $\lambda \in (0, \frac{1}{3}]$, the only issue left towards the global existence of Hölder continuous solution with exponent $1 - \lambda$ is that we cannot find the uniform L^∞ bound on two variables p and q defined in (3.9), which take similar role as ξ for (1.1). Hence, the issue whether the solution u is Hölder continuous with exponent $1 - \lambda$ is changed to another issue whether variables p and q in the semi-linear system are bounded away from zero and infinity, at the breakdown of classical solution. Especially, if we use numeric method to study these two issues, the latter one is much simpler than the first one.

Several numeric experiments are given in this paper, which all indicate that p and q are bounded away from zero and infinity even when classical solution of (1.2) breaks down. As a consequence, it is reasonable to expect that the solution $u(x, t)$ is Hölder continuous with exponent $1 - \lambda$ even when the gradient blowup happens in finite time, where $\lambda \in (0, \frac{1}{3}]$.

The rest of the paper is divided into two sections. In Section 2, we consider system (1.1) and prove Theorem 1. In Section 3, we will discuss the wave equation (1.2).

2. Wave in a unitary direction

In this section, we consider system (1.1) and prove Theorem 1.

In Subsection 2.1, we first define a new coordinate (Y, T). Then based on (1.1) and initial condition (1.9)(1.10), we derive a semi-linear system for several unknowns on new independent variables Y and T .

In Subsection (2.2), we prove existence and uniqueness of solution for the new semi-linear system on (Y, T) coordinates.

Finally, in Subsection (2.3), after making an inverse transformation on the constructed solution on (Y, T) coordinates, we recover a weak solution for (1.1) on the (x, t)-coordinates and complete the proof of Theorem 1.

2.1. **New coordinates.** In this subsection, we derive some equations valid for smooth solutions of (1.1). We denote the variables

$$\begin{cases} S := u_t + f'(u)u_x \\ R := u_x \,. \end{cases} \tag{2.1}$$

By (1.1), we have

$$\begin{cases} S_x & = & (1 - \lambda)\, f''(u)\, R^2 \\ R_t + f'(u)R_x & = & -\lambda f''(u)\, R^2 \,. \end{cases} \tag{2.2}$$

In order to reduce the equation (1.1) into a semi-linear system, it is convenient to change the independent variables. The equation of the characteristic is

$$\frac{dx^c(t)}{dt} = f'\big(u(x^c(t), t)\big). \tag{2.3}$$

We denote the characteristic passing through the point (x, t) as

$$a \mapsto x^c(a; x, t) \quad \text{or equivalently} \quad b \mapsto t^c(b; x, t),$$

where a and b are the time and space variables of the characteristic, respectively. Then we introduce new coordinates (Y, T), such that
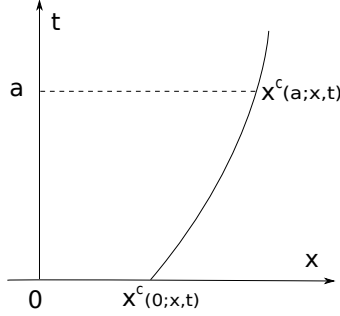

FIGURE 1. A characteristic $a \mapsto x^c(a; x, t)$ passing (x, t).

$$Y \equiv Y(x, t) := \begin{cases} \int_0^{x^c(0; x, t)} (1 + R^2(x', 0))^{\frac{1}{2\lambda}}\, dx', \\ \qquad \text{when the characteristic passing } (x, t) \text{ interacts } t = 0; \\ \\ -t^c(0; x, t)\, f'(0) \\ \qquad \text{when the characteristic passing } (x, t) \text{ interacts } x = 0, \end{cases} \tag{2.4}$$

with $(x, t) \in \mathbb{R}^+ \times \mathbb{R}^+$ and

$$T \equiv T(x, t) := t \,. \tag{2.5}$$

Clearly, Y is constant along a characteristic x^c by its definition. So

$$Y_t + f'(u)Y_x = 0, \quad T_t = 1 \quad \text{and} \quad T_x = 0. \tag{2.6}$$

Using (2.6), for any smooth function m, we have

$$\begin{cases} m_t + f'(u)m_x = m_Y\,(Y_t + f'(u)Y_x) + m_T\,(T_t + f'(u)T_x) = m_T, \\ m_x = m_Y\,Y_x + m_T\,T_x = m_Y\,Y_x. \end{cases} \tag{2.7}$$

Then we derive a semi-linear system on (Y,T)-coordinates. In order to complete the system, we introduce several new variables:

$$v := 2\arctan u_x \quad \text{and} \quad \xi := \frac{(1+R^2)^{\frac{1}{2\lambda}}}{Y_x}. \tag{2.8}$$

Hence

$$\frac{1}{1+R^2} = \cos^2 \frac{v}{2} \quad \text{and} \quad \frac{R}{1+R^2} = \frac{1}{2}\sin v. \tag{2.9}$$

By (2.2) and (2.7), we have

$$u_Y = \frac{u_x}{Y_x} = \frac{1}{2}\xi\,\sin v\,(\cos^2\frac{v}{2})^{\frac{1}{2\lambda}-1}, \tag{2.10}$$

$$v_T = \frac{2\,R_T}{1+R^2} = -2\lambda\,f''(u)\,\frac{R^2}{1+R^2} = -2\lambda\,f''(u)\sin^2\frac{v}{2}, \tag{2.11}$$

and also using (2.6), we have

$$\begin{aligned} \xi_T &= \quad (\frac{(1+R^2)^{\frac{1}{2\lambda}}}{Y_x})_T \\ &= \quad \frac{1}{\lambda}\frac{(1+R^2)^{\frac{1}{2\lambda}-1}\,R\,R_T}{Y_x} - \frac{(1+R^2)^{\frac{1}{2\lambda}}}{Y_x^2}(Y_x)_T \\ &= \quad -\frac{f''(u)(1+R^2)^{\frac{1}{2\lambda}-1}R^3}{Y_x} - \frac{(1+R^2)^{\frac{1}{2\lambda}}}{Y_x^2}\big((Y_t+f'(u)\,Y_x)_x - f''(u)RY_x\big) \\ &= \quad f''(u)\frac{(1+R^2)^{\frac{1}{2\lambda}}}{Y_x}\frac{R}{1+R^2} \\ &= \quad \tfrac{1}{2}f''(u)\,\xi\,\sin v. \end{aligned} \tag{2.12}$$

Summarizing (2.10)~(2.12), we have a semi-linear system:

$$\begin{cases} u_Y &= \tfrac{1}{2}\xi\,\sin v\,(\cos^2\frac{v}{2})^{\frac{1}{2\lambda}-1}, \\ v_T &= -2\lambda\,f''(u)\sin^2\frac{v}{2}, \\ \xi_T &= \tfrac{1}{2}f''(u)\,\xi\,\sin v. \end{cases} \tag{2.13}$$

Furthermore,

$$u_T = u_t + f'(u)\,u_x = S, \tag{2.14}$$

and

$$S_Y = \frac{S_x}{Y_x} = (1-\lambda)\,f''(u)\,\xi\,\sin^2\frac{v}{2}\,(\cos^2\frac{v}{2})^{\frac{1}{2\lambda}-1}. \tag{2.15}$$

Hence we have another semi-linear system

$$
\begin{cases}
u_T &= S\,, \\
S_Y &= (1-\lambda)\,f''(u)\,\xi\,\sin^2 \tfrac{v}{2}\,(\cos^2 \tfrac{v}{2})^{\frac{1}{2\lambda}-1}\,, \\
v_T &= -2\lambda\,f''(u)\sin^2 \tfrac{v}{2}\,, \\
\xi_T &= \tfrac{1}{2}f''(u)\,\xi\,\sin v\,.
\end{cases}
\tag{2.16}
$$

It is easy to see that semi-linear systems (2.13) and (2.16) are both invariant under translation by 2π in v. It would be more precise to use e^{iv} as variable. For simplicity, we use $v \in [-\pi, \pi]$ with endpoints identified.

2.2. Existence on the new coordinates.

In this subsection, we prove the existence of solution for (2.13) with initial and boundary data converted from (1.9) and (1.10). To avoid the confusion, the reader should be aware that in this section we solve variables (u, v, ξ) by system (2.13) when $\lambda \in (0, \tfrac{1}{2}]$, and we do not use the equations (1.1) and definition (2.8) except when we assign the initial and boundary data. We also use (2.16) when $\lambda \in (0, \tfrac{1}{3}] \cup \tfrac{1}{2}$ to show better regularity for the solution. We will recover a weak solution for (1.1) on the (x, t)-coordinates in subsection (2.3),

2.2.1. *The initial boundary value problem on new coordinates.* The initial lines $T = 0$ and $t = 0$ are the same line. By (2.4), the curve $Y := \Gamma_b(T)$ on the (Y, T)-plane transformed from $\{x = 0,\ t \geq 0\}$ is

$$
Y = \Gamma_b(T) = -f'(0)T\,.
\tag{2.17}
$$

Recall $f'(0) \geq 0$.

After transformation from (x, t)-coordinates to (Y, T)-coordinates, set $(\mathbb{R}^+, \mathbb{R}^+)$ changes to a new set named Ω:

$$
\Omega := \{(Y, T);\ Y \geq \Gamma_b(T),\ T \geq 0\}.
\tag{2.18}
$$

We consider initial boundary value problem of (2.13) on Ω with following initial and boundary data given by (1.9).

The initial data on $(Y, 0)$ with $Y \geq 0$ are

$$
\begin{cases}
u(Y, 0) &:= u_0\big(x(Y, 0)\big), \\
v(Y, 0) &:= 2\arctan(u_0'\big(x(Y, 0)\big)), \\
\xi(Y, 0) &:= 1.
\end{cases}
\tag{2.19}
$$

The boundary conditions on $Y = \Gamma_b(T)$ with $T \geq 0$ are

$$
\begin{cases}
u(\Gamma_b(T), T) &:= 0, \\
v(\Gamma_b(T), T) &:= 0, \\
\xi(\Gamma_b(T), T) &:= 1.
\end{cases}
\tag{2.20}
$$

Theorem 2. *Assume all conditions on initial and boundary data in Theorem 1 hold. Then the corresponding problem (2.13) with boundary data (2.19)(2.20) has a unique solution defined for all $(Y, T) \in \Omega$.*

Moreover, one has the stability of the solution. Assume that a sequence of C^1 functions u_0^k satisfy

$$u_0^k \to u_0 \,, \quad (u_0^k)_x \to (u_0)_x \,,$$

uniformly on any bounded subset of $\mathbb{R}^+$. Then one has the convergence of the corresponding solutions for (2.13):

$$(u^k, v^k, \xi^k) \to (u, v, \xi)$$

uniformly on bounded subsets of Ω.

Proof. The proof is based on the locally Lipschitz continuity of the right hand side of equations (2.13). Actually, on any bounded domain

$$\Omega_r := \{(Y, T); \ Y \geq \Gamma_b(T), \ 0 \leq T \leq r, \ Y \leq r\} \,.$$

Since $|f''(u)|$ is uniformly bounded above, using the last equation in (2.13), we could find a priori upper bound of ξ, i.e. $\xi < e^{\frac{1}{2} r \max_{u \in \Omega_r}(f''(u))}$. Then as long as $\lambda \in (0, \frac{1}{2}]$, the right hand side of equations (2.13) is Lipschitz continuous on (u, v, ξ) in Ω_r.

Introduce a space of functions

$$\Theta_r := \left\{ f : \Omega_r \mapsto \mathbb{R}; \ \|f\|_* := \underset{(Y,T) \in \Omega_r}{\text{ess sup}} \ e^{-\kappa(T+|Y|)} |f(Y, T)| < \infty \right\},$$

where κ is a suitably large constant. It is straightforward to construct a solution $(u, v, \xi)(Y, T)$ with $(Y, T) \in \Omega_r$ as a fixed point in $\Theta_r \times \Theta_r \times \Theta_r$, using the integral forms of (2.13) and the fact that the right hand side of (2.13) is Lipschitz. When $r \to \infty$, we can get a unique solution defined on the whole domain of Ω. As a consequence of the fixed point argument, solutions have stability in the sense of Theorem 2. We refer readers to [4] for more details.

□

By Theorem 2, we have the regularity of (u, v, ξ). In fact, the equation (2.13) implies, on any bounded set of Ω, when $\gamma \in (0, \frac{1}{2}]$:

- u is Lipschitz continuous w.r.t Y, measurable w.r.t T.
- v, ξ are Lipschitz continuous w.r.t T, measurable w.r.t Y.
- u, v, ξ have finite L^∞ norm, and $\xi > 0$.

When $\gamma \in (0, \frac{1}{3}] \cup \frac{1}{2}$, right hand side of (2.16) is also locally Lipschitz continuous, hence we could also prove global existence of the solutions using (2.16) by same argument. This solution is exactly the same solution of (2.13) because

$$u_{YT} = (1 - \lambda) f''(u) \, \xi \, \sin^2 \frac{v}{2} \, (\cos^2 \frac{v}{2})^{\frac{1}{2\lambda} - 1} = u_{TY} \,.$$

Hence we have

- u, S are Lipschitz continuous w.r.t Y, measurable w.r.t T.
- u, v, ξ are Lipschitz continuous w.r.t T, measurable w.r.t Y.
- u, v, S, ξ have finite L^∞ norm, and $\xi > 0$.

2.3. **Existence on (x, t)-coordinates.** The map $(Y, T) \mapsto (x, t)$ from Ω to $(\mathbb{R}^+, \mathbb{R}^+)$ can be constructed by the following procedure.

First we use $t = T$ and

$$x_Y = \xi(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}} \quad \text{and} \quad x_T = f'(u) \tag{2.21}$$

to do the inverse transformation from (Y, T) to (x, t). It is easy to check that the two equations in (2.21) are equivalent:

$$x_{TY} = \frac{1}{2} f''(u) \xi \, \sin v \, (\cos^2 \frac{v}{2})^{\frac{1}{2\lambda} - 1} = x_{YT} .$$

So we can recover the function $x(Y, T)$ by integrating either $x_Y = \xi(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}}$ or $x_T = f'(u)$. By (2.21), it is easy to recover (2.6) and (2.7).

For any smooth function m,

$$\xi(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}} m_x = m_x x_Y = m_Y . \tag{2.22}$$

By (2.21) and (2.22),

$$m_x \, dx \, dt = m_Y \, dY \, dT \quad \text{and} \quad dx \, dt = \xi(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}} \, dY \, dT , \tag{2.23}$$

and, for any t,

$$m_x \, dx = m_Y \, dY , \quad \text{and} \quad dx = \xi(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}} \, dY . \tag{2.24}$$

Then we define u as a function of the original variables (x, t) by

$$u(x, t) = u\big(Y(x, t), T(t)\big) .$$

Note the fact that the map $x(Y, T)$ may not be one-to-one does not cause any real difficulty. Indeed, given (x^*, t^*), we can choose an arbitrary Y^* such that $x(Y^*, t^*) = x^*$ and $T^* = t^*$, then define $u(x^*, t^*) = u(Y^*, T^*)$. To prove that the values of u do not depend on the choice of Y^*, we proceed as follows. Assume that there are two distinct points such that $x(Y_1, t^*) = x(Y_2, t^*) = x^*$, which shows that $x_Y(Y, t^*) = \xi(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}} = 0$ for $Y \in [Y_1, Y_2]$ because of the monotonicity of Y on x by (2.21). This shows $\cos(\frac{v}{2}) = 0$ for $Y \in [Y_1, Y_2]$, where recall $\xi > 0$. By (2.13), $u_Y = 0$ when $Y \in [Y_1, Y_2]$. Hence, we get $u(Y_1, t^*) = u(Y_2, t^*)$.

By (2.13), (2.7), (2.22), we could retrieve (2.8).

2.3.1. *Proof of Theorem 1.*

Proof. We first consider the regularity of the solution. For any given time t, by (2.22), (2.24) and (2.13),

$$\int_{x_1}^{x_2} |u_x|^{\frac{1}{\lambda}} \, dx = \int_{Y_1}^{Y_2} \Big| \frac{u_Y}{\xi(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}}} \Big|^{\frac{1}{\lambda}} \xi(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}} dY = \int_{Y_1}^{Y_2} |\sin \frac{v}{2}|^{\frac{1}{\lambda}} \xi \, dY < \infty,$$
$$\tag{2.25}$$

on any bounded interval $[x_1, x_2] \in \mathbb{R}^+$. Hence, for any time t, the solution $u(\cdot, t) \in W_{loc}^{1, \frac{1}{\lambda}}(\mathbb{R}^+)$.

Finally, we prove that the function u provides a weak solution of (1.1). By (2.13), (2.22), we have

$$\iint_{\mathbb{R}^+ \times \mathbb{R}^+} \left\{ -u_x \left(\phi_t + f'(u) \phi_x \right) + (\lambda - 1) f''(u)(u_x)^2 \phi \right\} \, dx \, dt$$

$$= \iint_{(Y,T) \in \Omega} \left\{ -u_Y \phi_T + (\lambda - 1) f''(u) \xi \sin^2 \frac{v}{2} (\cos^2 \frac{v}{2})^{\frac{1}{2\lambda} - 1} \phi \right\} \, dY \, dT$$

$$= \iint_{(Y,T) \in \Omega} \left\{ u_{YT} + (\lambda - 1) f''(u) \xi \sin^2 \frac{v}{2} (\cos^2 \frac{v}{2})^{\frac{1}{2\lambda} - 1} \right\} \phi \, dY \, dT$$

$$= \iint_{(Y,T) \in \Omega} \left\{ (\frac{1}{2} \xi \sin v \, (\cos^2 \frac{v}{2})^{\frac{1}{2\lambda} - 1})_T \right.$$

$$\left. + (\lambda - 1) f''(u) \xi \sin^2 \frac{v}{2} (\cos^2 \frac{v}{2})^{\frac{1}{2\lambda} - 1} \right\} \phi \, dY \, dT$$

$$= 0, \tag{2.26}$$

where $\phi(x,t) \in C_c^1(\mathbb{R}^+ \times \mathbb{R}^+)$.

When $0 < \lambda \le \frac{1}{3}$ and $\lambda = \frac{1}{2}$,

$$u_t + f'(u) \, u_x = S \in L^\infty.$$

Now we show regularities along two directions where recall another one is given in (2.25). By the Sobolev embedding theorem, solution $u(x,t)$ is locally Hölder continuous on both x and t with exponent $1 - \lambda$. So we complete the proof of Theorem 1. $\qquad \square$

3. Second order wave equations

For equation (1.2), by introducing new characteristic coordinates, we get a semi-linear system when $0 < \lambda < 1$. Using this semi-linear system, we discuss the regularity of the solutions for (1.2).

3.1. A semi-linear system on new coordinates. In this section, we only consider the smooth solution, and derive a semi-linear system from the smooth solution of (1.2).

First, we introduce the new coordinate. We define

$$R := u_t + c(u)u_x, \qquad S := u_t - c(u)u_x. \tag{3.1}$$

So we have

$$u_t = \frac{R + S}{2} \quad \text{and} \quad u_x = \frac{R - S}{2c}.$$

To keep the tradition, we still use R here to denote the gradient variables. By (1.2), we have

$$\begin{cases} R_t - cR_x & = \frac{c'}{4c} \left[2\lambda R^2 + (2\lambda - 2)S^2 - 2(2\lambda - 1)RS \right] \\ S_t + cS_x & = \frac{c'}{4c} \left[2\lambda S^2 + (2\lambda - 2)R^2 - 2(2\lambda - 1)RS \right] \end{cases} \tag{3.2}$$

where $c' = \frac{d}{du}c(u)$. We define the forward and backward characteristics passing the point (x, t) as follows

$$\begin{cases} \dfrac{d}{ds}x^{\pm}(s; x, t) = \pm c(u(s, x^{\pm}(s; x, t))), \\ x^{\pm}|_{s=t} = x. \end{cases} \tag{3.3}$$

Along the characteristics, we define the new coordinate:

$$X := \int_0^{x^-(0;x,t)} [1 + R^2(y, 0)]^{\frac{1}{2\lambda}}\, dy \quad \text{and} \quad Y := \int_{x^+(0;x,t)}^0 [1 + S^2(y, 0)]^{\frac{1}{2\lambda}}\, dy\,.$$

This implies

$$X_t - c(u)X_x = 0, \quad Y_t + c(u)Y_x = 0\,. \tag{3.4}$$

Here, again without ambiguity, we still use Y as a new coordinate for (1.2). For any smooth function m, we obtain by using (3.4) that

$$\begin{aligned} m_t + c(u)m_x &= (X_t + c(u)X_x)m_X = 2c(u)X_x m_X \\ m_t - c(u)m_x &= (Y_t - c(u)Y_x)m_Y = -2c(u)Y_x m_Y. \end{aligned} \tag{3.5}$$

Next, we introduce some new variables for the semi-linear system. Without ambiguity, we still use v to denote an unknown in the semi-linear system in this section. So the reader can easily compare the semi-linear systems for (1.1) and (1.2). We define

$$w := 2\arctan R \quad \text{and} \quad v := 2\arctan S, \tag{3.6}$$

so we have

$$\frac{1}{1 + R^2} = \cos^2\frac{w}{2}, \qquad \frac{R}{1 + R^2} = \frac{1}{2}\sin w, \tag{3.7}$$

and

$$\frac{1}{1 + S^2} = \cos^2\frac{v}{2}, \qquad \frac{S}{1 + S^2} = \frac{1}{2}\sin v. \tag{3.8}$$

Then, we define

$$p := \frac{(1 + R^2)^{\frac{1}{2\lambda}}}{X_x} \quad \text{and} \quad q := \frac{(1 + S^2)^{\frac{1}{2\lambda}}}{-Y_x}. \tag{3.9}$$

Finally, after having the new coordinates and new variables, we calculate the equations for u, w, v, p and q. By (3.1), (3.2), (3.5) and (3.7)$\sim$(3.9), one get the equations for u:

$$u_X = \frac{u_t + cu_x}{2cX_x} = \frac{1}{2}\sin w\,(\cos^2\frac{w}{2})^{\frac{1}{2\lambda}-1}p, \tag{3.10}$$

$$u_Y = \frac{u_t - cu_x}{-2cY_x} = \frac{1}{2}\sin v\,(\cos^2\frac{v}{2})^{\frac{1}{2\lambda}-1}q. \tag{3.11}$$

Then, from (3.2)

$$\begin{aligned} w_t - cw_x &= \frac{2}{1 + R^2}(R_t - cR_x) \\ &= \frac{c'}{c}\frac{1}{1 + R^2}\left[\lambda R^2 + (\lambda - 1)S^2 - (2\lambda - 1)RS\right] \end{aligned}$$

so by (3.1), (3.2), (3.5) and (3.7)~(3.9), one has

$$
\begin{aligned}
w_Y &= \frac{w_t - cw_x}{-2cY_x} \\
&= \frac{c'}{2c^2}\, q\, \frac{1}{(1+S^2)^{\frac{1}{2\lambda}}}\, \frac{1}{1+R^2}\left[\lambda R^2 + (\lambda - 1)S^2 - (2\lambda - 1)RS\right] \\
&= \frac{c'}{2c^2}\, q\, (\cos^2\tfrac{v}{2})^{\frac{1}{2\lambda}-1}\Big[\lambda \sin^2\tfrac{w}{2}\cos^2\tfrac{v}{2} + (\lambda - 1)\sin^2\tfrac{v}{2}\cos^2\tfrac{w}{2} \\
&\qquad\qquad\qquad\qquad -\frac{2\lambda - 1}{4}\sin w \sin v\Big]\,.
\end{aligned}
\tag{3.12}
$$

Similarly, one has

$$
\begin{aligned}
v_t + cv_x &= \frac{2}{1+S^2}(S_t + cS_x) \\
&= \frac{c'}{c}\,\frac{1}{1+S^2}\left[\lambda S^2 + (\lambda - 1)R^2 - (2\lambda - 1)RS\right]
\end{aligned}
$$

and

$$
\begin{aligned}
v_X &= \frac{v_t + cv_x}{2cX_x} \\
&= \frac{c'}{2c^2}\, p\, \frac{1}{(1+R^2)^{\frac{1}{2\lambda}}}\, \frac{1}{1+R^2}\left[\lambda S^2 + (\lambda - 1)R^2 - (2\lambda - 1)RS\right] \\
&= \frac{c'}{2c^2}\, p\, (\cos^2\tfrac{w}{2})^{\frac{1}{2\lambda}-1}\Big[\lambda \sin^2\tfrac{v}{2}\cos^2\tfrac{w}{2} + (\lambda - 1)\sin^2\tfrac{w}{2}\cos^2\tfrac{v}{2} \\
&\qquad\qquad\qquad\qquad -\frac{2\lambda - 1}{4}\sin w \sin v\Big]\,.
\end{aligned}
\tag{3.13}
$$

Lastly, we derive equations for p and q. By (3.4),

$$
X_{xt} - cX_{xx} = (X_t - cX_x)_x + c'u_x X_x = \frac{c'}{2c}(R - S)X_x\,.
$$

So, by (3.1), (3.2), (3.5) and (3.7)~(3.9), one has

$$
\begin{aligned}
p_t - cp_x &= (\frac{(1+R^2)^{\frac{1}{2\lambda}}}{X_x})_t - c(\frac{(1+R^2)^{\frac{1}{2\lambda}}}{X_x})_x \\
&= \frac{(1+R^2)^{\frac{1}{2\lambda}-1}}{\lambda X_x}R[R_t - cR_x] - \frac{(1+R^2)^{\frac{1}{2\lambda}}}{X_x^2}(X_{xt} - cX_{xx}) \\
&= \frac{c'}{2c}\frac{(1+R^2)^{\frac{1}{2\lambda}-1}}{X_x}\left[\frac{\lambda - 1}{\lambda}RS^2 - \frac{\lambda - 1}{\lambda}R^2 S - R + S\right],
\end{aligned}
$$

hence

$$
\begin{aligned}
p_Y &= \frac{p_t - cp_x}{-2cY_x} \\
&= \frac{c'}{4c^2}\, pq\, \frac{1}{(1+S^2)^{\frac{1}{2\lambda}}}\, \frac{1}{1+R^2}\left[\frac{\lambda-1}{\lambda}RS^2 - \frac{\lambda-1}{\lambda}R^2S - R + S\right] \\
&= \frac{c'}{8c^2}\, pq\, (\cos^2\frac{v}{2})^{\frac{1}{2\lambda}-1}\left[\frac{\lambda-1}{\lambda}\sin w\, \sin^2\frac{v}{2} - \frac{\lambda-1}{\lambda}\sin v\, \sin^2\frac{w}{2}\right. \\
&\qquad \left. - \sin w\, \cos^2\frac{v}{2} + \sin v\, \cos^2\frac{w}{2}\right].
\end{aligned} \tag{3.14}
$$

Similarly, by (3.4),

$$
Y_{xt} + cY_{xx} = (Y_t + cY_x)_x - c'u_x Y_x = -\frac{c'}{2c}(R-S)Y_x\,.
$$

Then, by (3.1), (3.2), (3.5) and (3.7)~(3.9), one has

$$
\begin{aligned}
q_t + cq_x &= (\frac{(1+S^2)^{\frac{1}{2\lambda}}}{-Y_x})_t + c(\frac{(1+S^2)^{\frac{1}{2\lambda}}}{-Y_x})_x \\
&= \frac{(1+S^2)^{\frac{1}{2\lambda}-1}}{-\lambda Y_x}S[S_t + cS_x] + \frac{(1+S^2)^{\frac{1}{2\lambda}}}{Y_x^2}(Y_{xt} + cY_{xx}) \\
&= -\frac{c'}{2c}\frac{(1+S^2)^{\frac{1}{2\lambda}-1}}{-Y_x}\left[\frac{\lambda-1}{\lambda}RS^2 - \frac{\lambda-1}{\lambda}R^2S - R + S\right],
\end{aligned}
$$

hence

$$
\begin{aligned}
q_X &= \frac{q_t + cq_x}{2cX_x} \\
&= -\frac{c'}{4c^2}\, pq\, \frac{1}{(1+R^2)^{\frac{1}{2\lambda}}}\, \frac{1}{1+S^2}\left[\frac{\lambda-1}{\lambda}RS^2 - \frac{\lambda-1}{\lambda}R^2S - R + S\right] \\
&= -\frac{c'}{8c^2}\, pq\, (\cos^2\frac{w}{2})^{\frac{1}{2\lambda}-1}\left[\frac{\lambda-1}{\lambda}\sin w\, \sin^2\frac{v}{2} - \frac{\lambda-1}{\lambda}\sin v\, \sin^2\frac{w}{2}\right. \\
&\qquad\qquad\qquad \left. - \sin w\, \cos^2\frac{v}{2} + \sin v\, \cos^2\frac{w}{2}\right].
\end{aligned} \tag{3.15}
$$

In conclusion, by (3.10)~(3.15) we have a semi-linear system

$$
\left\{
\begin{array}{rcl}
u_X &=& \frac{1}{2}\sin w \,(\cos^2 \frac{w}{2})^{\frac{1}{2\lambda}-1} p \\[2mm]
u_Y &=& \frac{1}{2}\sin v \,(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}-1} q \\[2mm]
w_Y &=& \frac{c'}{2c^2}\, q \,(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}-1}\big[\lambda \sin^2 \frac{w}{2}\cos^2 \frac{v}{2} + (\lambda-1)\sin^2 \frac{v}{2}\cos^2 \frac{w}{2} \\[2mm]
&& \hspace{5cm} - \frac{2\lambda-1}{4}\sin w \sin v\big] \\[2mm]
v_X &=& \frac{c'}{2c^2}\, p \,(\cos^2 \frac{w}{2})^{\frac{1}{2\lambda}-1}\big[\lambda \sin^2 \frac{v}{2}\cos^2 \frac{w}{2} + (\lambda-1)\sin^2 \frac{w}{2}\cos^2 \frac{v}{2} \\[2mm]
&& \hspace{5cm} - \frac{2\lambda-1}{4}\sin w \sin v\big] \\[2mm]
p_Y &=& \frac{c'}{8c^2}\, pq \,(\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}-1}\big[\frac{\lambda-1}{\lambda}\sin w \sin^2 \frac{v}{2} - \frac{\lambda-1}{\lambda}\sin v \sin^2 \frac{w}{2} \\[2mm]
&& \hspace{5cm} - \sin w \cos^2 \frac{v}{2} + \sin v \cos^2 \frac{w}{2}\big] \\[2mm]
q_X &=& -\frac{c'}{8c^2}\, pq \,(\cos^2 \frac{w}{2})^{\frac{1}{2\lambda}-1}\big[\frac{\lambda-1}{\lambda}\sin w \sin^2 \frac{v}{2} - \frac{\lambda-1}{\lambda}\sin v \sin^2 \frac{w}{2} \\[2mm]
&& \hspace{5cm} - \sin w \cos^2 \frac{v}{2} + \sin v \cos^2 \frac{w}{2}\big].
\end{array}
\right.
\tag{3.16}
$$

3.2. **Regularity of solutions when $\lambda \in (0, \frac{1}{3}]$.** In the rest of this paper, we restrict our consideration on the case when $\lambda \in (0, \frac{1}{3}]$. Recall that $c(u)$ is assumed to be uniformly positive and bounded.

If we first suppose that there is a L^∞ bound on p and q on any bounded set of the (X, Y)-plane (on $t \geq 0$ part), then the right hand side of semi-linear system (3.16) is Lipschitz on u, v, w, p and q when $\lambda \in (0, \frac{1}{3}]$ on any bounded set of the (X, Y)-plane (on $t \geq 0$ part).

Hence, using similar method as those used for (1.1) in Theorem 1 and for (1.6) in [4], we could prove the global existence of solution for semi-linear system (3.16) then recover a weak solution for (1.2), where $u_t \pm c(u)u_x$ are in $W^{1, \frac{1}{\lambda}}$, because we assume that p and q are bounded. Hence, the solution u is Hölder continuous with exponent $1 - \lambda$. We leave the details to the reader and also refer the reader to [4].

Unfortunately, till now, we still have no method to bound p and q, hence the existence and regularity of the solution is still a conjecture and only can be supported by numerical experiments which will be shown later. For (1.6), the bounds on p and q are found in [4] by exploring energy law. But equation (1.2) with $\lambda \neq \frac{1}{2}$ is not endowed with such an energy law. Instead we have an equation

$$
(\cos^2 \frac{w}{2})^{\frac{1}{2\lambda}-1} p_Y + (\cos^2 \frac{v}{2})^{\frac{1}{2\lambda}-1} q_X = 0. \tag{3.17}
$$

On the other hand, system (3.16) still can help us analyze the original equation (1.2). As we discussed, to show whether u is Hölder continuous with exponent $1-\lambda$, by numerical experiments, we only have to test whether p and q remain uniformly positive and bounded when v or w attains π, i.e. when gradient blowup happens.

Recall that the region $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ is transformed to the region $\{Y \geq \phi(X)\}$ in (X, Y) plane, where $t = 0$ is corresponding to the curve $Y = \phi(X)$

with $\phi(X)$ strictly decreasing on X. Then we do some numerical experiments on the bounded domain of

$$\bar{\Omega}_r = \{(X,Y); \ Y \geq \phi(X), \quad X \leq r, \quad Y \leq r\}.$$

We can iteratively find the solution by initially set

$$\begin{cases} u_0(X,Y) &=& u(\phi^{-1}(Y),Y) \\ w_0(X,Y) &=& w(X,\phi(X)) \\ v_0(X,Y) &=& v(\phi^{-1}(Y),Y) \\ p_0(X,Y) &=& p(X,\phi(X)) \\ q_0(X,Y) &=& q(\phi^{-1}(Y),Y). \end{cases} \tag{3.18}$$

with $u(\phi^{-1}(Y),Y)$ given by $u(x,t=0) = sech(x)$. We also set $u_t(x,t = 0) = u_x(x,t=0)$ initially and use $c(u) = \sqrt{\cos^2(u)+1}$. Then we do the iteration according to the following integral equations coming from (3.16).

$$\begin{cases} u_{n+1}(X,Y) = u_n(\phi^{-1}(Y),Y) + \int_{\phi^{-1}(Y)}^{X} \frac{1}{2}\sin w_n \, (\cos^2 \frac{w_n}{2})^{\frac{1}{2\lambda}-1} p_n dX \\ w_{n+1}(X,Y) = w_n(X,\phi(X)) + \int_{\phi(X)}^{Y} \frac{c'}{2c^2} q \, (\cos^2 \frac{v_n}{2})^{\frac{1}{2\lambda}-1} \big[\lambda \sin^2 \frac{v_n}{2} \cos^2 \frac{w_n}{2} \\ \qquad\qquad +(\lambda-1)\sin^2 \frac{v_n}{2} \cos^2 \frac{w_n}{2} - \frac{2\lambda-1}{4}\sin w \sin v \big] dY \\ v_{n+1}(X,Y) = v_n(\phi^{-1}(Y),Y) + \int_{\phi^{-1}(Y)}^{X} \frac{c'}{2c^2} p_n \, (\cos^2 \frac{w_n}{2})^{\frac{1}{2\lambda}-1} \big[\lambda \sin^2 \frac{v_n}{2} \cos^2 \frac{w_n}{2} \\ \qquad\qquad +(\lambda-1)\sin^2 \frac{w_n}{2} \cos^2 \frac{v_n}{2} - \frac{2\lambda-1}{4}\sin w_n \sin v_n \big] dX \\ p_{n+1}(X,Y) = p_n(X,\phi(X)) + \int_{\phi(X)}^{Y} \frac{c'}{8c^2} p_n q_n \, (\cos^2 \frac{v_n}{2})^{\frac{1}{2\lambda}-1} \big[\frac{\lambda-1}{\lambda}\sin w_n \, \sin^2 \frac{v_n}{2} \\ \qquad\qquad -\frac{\lambda-1}{\lambda}\sin v_n \, \sin^2 \frac{w_n}{2} - \sin w_n \, \cos^2 \frac{v_n}{2} + \sin v_n \, \cos^2 \frac{w_n}{2} \big] dY \\ q_{n+1}(X,Y) = q_n(\phi^{-1}(Y),Y) - \int_{\phi^{-1}(Y)}^{X} \frac{c'}{8c^2} p_n q_n \, (\cos^2 \frac{w_n}{2})^{\frac{1}{2\lambda}-1} \big[\frac{\lambda-1}{\lambda}\sin w_n \sin^2 \frac{v_n}{2} \\ \qquad\qquad -\frac{\lambda-1}{\lambda}\sin v_n \, \sin^2 \frac{w_n}{2} - \sin w_n \, \cos^2 \frac{v_n}{2} + \sin v_n \, \cos^2 \frac{w_n}{2} \big] dX. \end{cases} \tag{3.19}$$

In a numeric experiment shown in Figure 2 when $\lambda = 1/4$, we see that p and q remain uniformly positive and bounded when w attains π, which strongly indicates that we could construct a solution $u(x,t)$ which is Hölder continuous with exponent $1-\lambda$, including finite time blowup (w attains π).

In fact, on the bounded set $\bar{\Omega}_r$ if p and q have positive upper and lower bounds, then equation (3.19) gives a contract mapping in

$$\Lambda_r := \left\{ f : \bar{\Omega}_r \mapsto \mathbb{R}; \ \|f\|_* := \text{ess} \sup_{(X,Y)\in\bar{\Omega}_r} e^{-\kappa(|X|+|Y|)}|f(X,Y)| < \infty \right\},$$

with sufficiently large κ. Hence u_n, v_n, w_n, p_n, q_n converge to a solution of equation (3.16). Then we could construct a solution $u(x,t)$ for (1.2) which is Hölder continuous with exponent $1-\lambda$, through similar inverse transformation as Theorem 1 and the one in [4] for (1.6).

Similar experiment has been done for $\lambda = 1/3$, which is shown in Figure 3. In this case, we can draw a same conclusion as case $\lambda = 1/4$.

The advantage of doing numeric analysis on the semi-linear system (3.16) instead of analyzing (1.2) directly is that we change a problem for Hölder

continuity for u when gradient blowup happens into a problem for L^∞ bounds of p and q when v or w attains π. Clearly, it is much easier to check the latter one than the first one by numerical methods. The semi-linear system is also easier to cope with than the quasi-linear system.
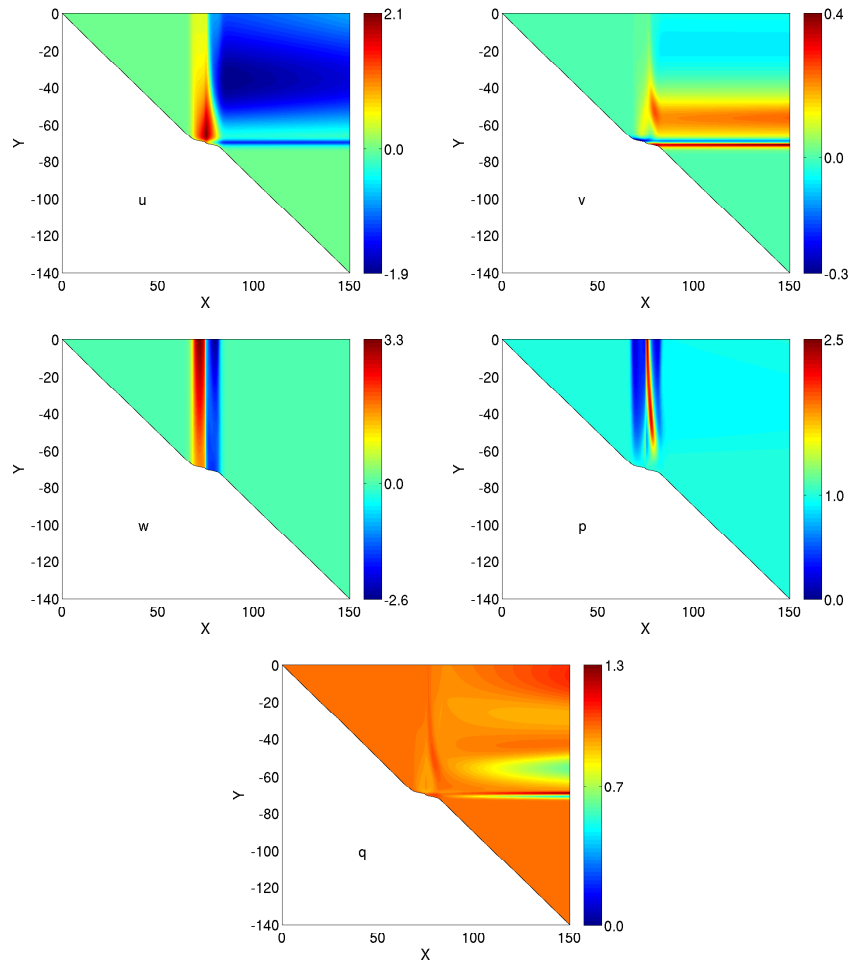

FIGURE 2. u, v, w, p, q in (X, Y) coordinate for $\lambda = \frac{1}{4}$. In this example, w attains π in finite time, but p and q are both uniformly bounded and positive.

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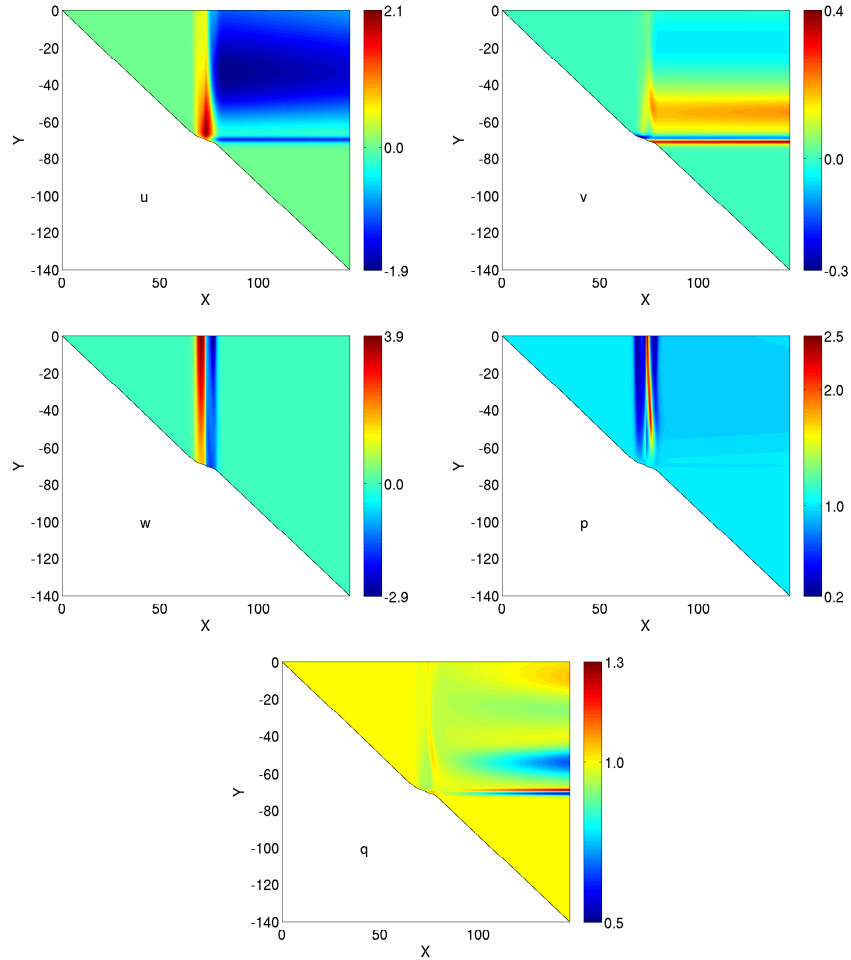
FIGURE 3. u, v, w, p, q in (X, Y) coordinate for $\lambda = \frac{1}{3}$. In this example, w attains π in finite time, but p and q are both uniformly bounded and positive.

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