ENERGY CONSERVATIVE SOLUTIONS TO A NONLINEAR WAVE SYSTEM OF NEMATIC LIQUID CRYSTALS

GENG CHEN, PING ZHANG, AND YUXI ZHENG

Abstract. We establish the global existence of solutions to the Cauchy problem for a system of hyperbolic partial differential equations in one space dimension modeling a type of nematic liquid crystals that has equal splay and twist coefficients. Our results have no restrictions on the angles of the director, as we use the director in its natural three-component form, rather than the two-component form of spherical angles.

1. Introduction

In this paper, we consider the global existence of energy conservative weak solutions to the following one-dimensional wave system of liquid crystals

$$\left\{\begin{array}{l}\partial_{tt}n_1-\partial_x(c^2(n_1)\partial_x n_1)=\big(-|\mathbf{n}_t|^2+(2c^2-\gamma)|\mathbf{n}_x|^2\big)n_1,\\ \partial_{tt}n_2-\partial_x(c^2(n_1)\partial_x n_2)=\big(-|\mathbf{n}_t|^2+(2c^2-\alpha)|\mathbf{n}_x|^2\big)n_2,\\ \partial_{tt}n_3-\partial_x(c^2(n_1)\partial_x n_3)=\big(-|\mathbf{n}_t|^2+(2c^2-\alpha)|\mathbf{n}_x|^2\big)n_3,\end{array}\right. \tag{1.1}$$

together with initial data

$$n_i|_{t=0}=n_{i0}\in H^1,\quad (n_i)_t|_{t=0}=n_{i1}\in L^2,\quad i=1,2,3, \tag{1.2}$$

where c depends on n_1 with $c^2(n_1)=\alpha+(\gamma-\alpha)n_1^2, \alpha>0, \gamma>0$, and $\mathbf{n}=(n_1,n_2,n_3)$ with

$$|\mathbf{n}|=1. \tag{1.3}$$

Subscripts t or x represent partial derivatives with respect to t or x, and $\partial_{tt}=\partial_t\partial_t$.

We firstly mention briefly the origin of this system. The mean orientation of the long molecules in a nematic liquid crystal is described by a director field of unit vectors, $\mathbf{n}\in\mathbb{S}^2$, the unit sphere. Associated with the director field $\mathbf{n}$, there is the well-known Oseen-Franck potential energy density W given by

$$W(\mathbf{n},\nabla\mathbf{n})=\frac{1}{2}\alpha(\nabla\cdot\mathbf{n})^2+\frac{1}{2}\beta(\mathbf{n}\cdot\nabla\times\mathbf{n})^2+\frac{1}{2}\gamma\,|\mathbf{n}\times(\nabla\times\mathbf{n})|^2. \tag{1.4}$$

The positive constants α, β, and γ are elastic constants of the liquid crystal, corresponding to splay, twist, and bend, respectively. For the special case $\alpha=\beta=\gamma$, the potential energy density reduces to $W(\mathbf{n},\nabla\mathbf{n})=\frac{1}{2}\alpha\,|\nabla\mathbf{n}|^2$, which is the potential energy density used in harmonic maps into $\mathbb{S}^2$. There are many studies on the constrained elliptic system of equations for $\mathbf{n}$ derived through variational principles from the potential (1.4), and on the parabolic flow associated with it, see [2, 5, 6, 8, 9, 17] and references therein. In the regime in which inertia effects dominate viscosity, however, the propagation of the orientation waves in the director field may then be modelled by the least action principle ([10, 1])

$$\frac{\delta}{\delta\mathbf{n}}\int_{\mathbb{R}^4}\left\{\frac{1}{2}\partial_t\mathbf{n}\cdot\partial_t\mathbf{n}-W(\mathbf{n},\nabla\mathbf{n})\right\}d\mathbf{x}\,dt=0,\qquad \mathbf{n}\cdot\mathbf{n}=1. \tag{1.5}$$

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In the special case $\alpha=\beta=\gamma$, this variational principle (1.5) yields the equation for harmonic wave maps from (1+3)-dimensional Minkowski space into $\mathbb{S}^2$, see [4, 11, 12] for example. One may also check [13, 14, 15, 16] for wave maps in dimension $(1+2)$. For planar deformations depending on a single space variable x, i.e, the director field has the special form

$$\mathbf{n}=\cos u(x,t)\mathbf{e}_x+\sin u(x,t)\mathbf{e}_y,$$

where the dependent variable $u\in\mathbb{R}^1$ measures the angle of the director field to the x-direction, and $\mathbf{e}_x$ and $\mathbf{e}_y$ are the coordinate vectors in the x and y directions, respectively, one finds that the variational principle (1.5) yields

$$u_{tt}-c_1(c_1u_x)_x=0 \tag{1.6}$$

with the wave speed c_1 given specifically by

$$c_1^2(u)=\gamma\cos^2 u+\alpha\sin^2 u. \tag{1.7}$$

If we let $\mathbf{n}$ be arbitrary in $\mathbb{S}^2$ while maintaining dependence on a single space variable x, i.e.,

$$\mathbf{n}=(\cos u,\sin u\cos v,\sin u\sin v) \tag{1.8}$$

where (u,v) are both functions of (x,t), we find that the Lagrangian density of (1.5) is

$$\frac{1}{2}\partial_t\mathbf{n}\cdot\partial_t\mathbf{n}-W\left(\mathbf{n},\nabla\mathbf{n}\right)=\frac{1}{2}\left[u_t^2-c_1^2(u)u_x^2\right]+\frac{1}{2}a^2(u)\left[v_t^2-c_2^2(u)v_x^2\right] \tag{1.9}$$

where

$$c_1^2(u)=\gamma\cos^2 u+\alpha\sin^2 u,\ \ c_2^2(u)=\gamma\cos^2 u+\beta\sin^2 u,\ \ a^2(u)=\sin^2 u. \tag{1.10}$$

The Euler-Lagrange equations are

$$\left\{\begin{array}{rcl} u_{tt}-c_1(u)(c_1(u)u_x)_x &=& aa'(v_t^2-c_2^2v_x^2)-a^2c_2c_2'v_x^2,\\ (a^2v_t)_t-(c_2^2a^2v_x)_x &=& 0. \end{array}\right. \tag{1.11}$$

See Alì and Hunter [1] for more details on the derivation of the above system.

Under the additional assumption that $c_1'(u)\geq 0$, Zhang and Zheng established the global existence of energy dissipative weak solutions to (1.6) in [18, 19]. Without any additional assumption, Bressan and Zheng established the global existence of energy conservative weak solutions to (1.6) in [3]. In [20, 21], Zhang and Zheng solved the global existence of energy conservative solutions to (1.11) under the un-physical assumption that $a(u)\geq a_{\min}>0$ (and $c_2<c_1$ for [21]). The goal of this paper is to get rid of this assumption in [20]. Indeed (1.5) can be equivalently reformulated as

$$\frac{\delta}{\delta\mathbf{n}}\int_{\mathbb{R}^2}\left\{\frac{1}{2}\partial_t\mathbf{n}\cdot\partial_t\mathbf{n}-W\left(\mathbf{n},\partial_x\mathbf{n}\right)+\frac{\lambda}{2}(|\mathbf{n}|^2-1)\right\}dx\,dt=0,$$

which gives rise to the Euler-Langrage equations

$$\partial_{tt}n_i+\partial_{n_i}W(\mathbf{n},\partial_x\mathbf{n})-\partial_x\left[\partial_{\partial_x n_i}W(\mathbf{n},\partial_x\mathbf{n})\right]=\lambda n_i,\quad\text{for}\quad i=1,2,3. \tag{1.12}$$

Using $|\mathbf{n}|=1$, multiplying (1.12) with n_i, and summing them up for i from 1 to 3, we obtain

$$\lambda=\sum_{i=1}^{3}\Big\{-|\partial_t n_i|^2+n_i\partial_{n_i}W(\mathbf{n},\partial_x\mathbf{n})-n_i\partial_x\left[\partial_{\partial_x n_i}W(\mathbf{n},\partial_x\mathbf{n})\right]\Big\}. \tag{1.13}$$

It is easy to calculate that (1.12) has the following energy conservation law:

$$\partial_t\big[\frac{1}{2}|\mathbf{n}_t|^2+W(\mathbf{n},\partial_x\mathbf{n})\big]-\partial_x\big[\sum_{i=1}^{3}\partial_t n_i\partial_{\partial_x n_i}W(\mathbf{n},\partial_x\mathbf{n})\big]=0. \tag{1.14}$$

On the other hand, we use (1.4) in the one-dimensional case to obtain

$$W(\mathbf{n},\partial_x\mathbf{n}) = \frac{\alpha}{2}(\partial_x n_1)^2 + \frac{\beta}{2}\big[(\partial_x n_2)^2 + (\partial_x n_3)^2\big] + \frac{1}{2}(\gamma-\beta)n_1^2|\mathbf{n}_x|^2, \tag{1.15}$$

from which and (1.13), we infer

$$\lambda = -|\mathbf{n}_t|^2 + \big(\beta + 2(\gamma-\beta)n_1^2\big)|\mathbf{n}_x|^2 + (\beta-\alpha)n_1\partial_x^2 n_1. \tag{1.16}$$

Combining (1.12), (1.15) and (1.16), we have

$$\partial_{tt}n_1 - \partial_x\big[c_1^2(n_1)\partial_x n_1\big] = \big\{-|\mathbf{n}_t|^2 + (2c_2^2-\gamma)|\mathbf{n}_x|^2 + 2(\alpha-\beta)(\partial_x n_1)^2\big\}n_1 \tag{1.17}$$

with

$$c_1^2(n_1) \overset{\text{def}}{=} \alpha + (\gamma-\alpha)n_1^2 \quad \text{and} \quad c_2^2(n_1) = \beta + (\gamma-\beta)n_1^2.$$

Similarly, we have

$$\begin{aligned} \partial_{tt}n_2 - \partial_x\big[c_2^2(n_1)\partial_x n_2\big] &= \big\{-|\mathbf{n}_t|^2 + (2c_2^2-\beta)|\mathbf{n}_x|^2 + (\beta-\alpha)n_1\partial_{xx}n_1\big\}n_2, \\ \partial_{tt}n_3 - \partial_x\big[c_2^2(n_1)\partial_x n_3\big] &= \big\{-|\mathbf{n}_t|^2 + (2c_2^2-\beta)|\mathbf{n}_x|^2 + (\beta-\alpha)n_1\partial_{xx}n_1\big\}n_3. \end{aligned} \tag{1.18}$$

In particular, taking $\alpha=\beta$ in (1.17) and (1.18) leads to (1.1).

When $\alpha=\beta$, the two speeds are equal, so we let

$$c^2(n_1) = c_1^2(n_1) = c_2^2(n_1) = \alpha + (\gamma-\alpha)n_1^2, \tag{1.19}$$

and it follows that

$$0 < \min\{\alpha,\gamma\} \le c^2(n_1) \le \max\{\alpha,\gamma\} < \infty; \quad \max\{|c'(n_1)|\} < \infty, \quad \text{for any} \quad |n_1| \le 1. \tag{1.20}$$

The energy equation (1.14) becomes

$$\frac{1}{2}\partial_t\big[|\mathbf{n}_t|^2 + c^2(n_1)|\mathbf{n}_x|^2\big] - \partial_x\big[c^2(n_1)\mathbf{n}_t\cdot\mathbf{n}_x\big] = 0, \tag{1.21}$$

where the energy density W in (1.15) becomes

$$W(\mathbf{n},\partial_x\mathbf{n}) = \frac{1}{2}c^2(n_1)|\partial_x\mathbf{n}|^2. \tag{1.22}$$

We consider in this paper the global existence of energy-conservative weak solutions of (1.1)(1.3) to its initial value problem with data (1.2). Since some smooth initial data for equation (1.6) is known to form singularity later in time, it is easy to see that the same is true for system (1.11) and (1.1). Thus we shall give up classical solutions and consider weak solutions instead.

Definition 1.1 (Weak solution)**.** *The vector function* $\mathbf{n}(t,x)$, *defined for all* $(t,x)\in\mathbb{R}\times\mathbb{R}$, *is a* **weak solution** *to the Cauchy problem* (1.1)∼(1.3) *if it satisfies*

- **(i):** *In the t-x plane, the functions* (n_1,n_2,n_3) *are locally Hölder continuous with exponent* $1/2$. *This solution* $t\mapsto(n_1,n_2,n_3)(t,\cdot)$ *is continuously differentiable as a map with values in* L^p_{loc}, *for all* $1\le p<2$. *Moreover, it is Lipschitz continuous with respect to (w.r.t.) the* L^2 *distance, i.e.*

$$\big\|n_i(t,\cdot) - n_i(s,\cdot)\big\|_{L^2} \le L\,|t-s|, \quad i = 1\sim 3, \tag{1.23}$$

for all $t,s\in\mathbb{R}$.

- **(ii):** *The functions* (n_1,n_2,n_3) *take on the initial conditions in (1.2) pointwise, while their temporal derivatives hold in* L^p_{loc} *for* $p\in[1,2[$.
- **(iii):** *The equations* (1.1) *hold in distributional sense for test functions* $\phi\in C^1_c(\mathbb{R}\times\mathbb{R})$.

Our conclusions are as follows.

Theorem 1.1 (Existence)**.** *The problem* (1.1)∼(1.3) *has a global weak solution defined for all* $(t,x) \in [0,\infty) \times \mathbb{R}$.

We shall use the method of energy-dependent coordinates, used in papers [3] and related Camassa-Holm equation, to dilate the singularity in building the solution. Our constructive procedure yields solutions which depend continuously on the initial data. Moreover, the "energy"

$$\mathcal{E}(t) := \frac{1}{2}\int \Big[|\mathbf{n}_t|^2 + c^2(n_1)|\mathbf{n}_x|^2\Big]\,dx \tag{1.24}$$

remains uniformly bounded by its initial level:

$$\mathcal{E}_0 \;:=\; \frac{1}{2}\int \Big[|\mathbf{n}_1(x)|^2 + c^2(n_{10}(x))|(\mathbf{n}_0(x))_x|^2\Big]\,dx. \tag{1.25}$$

More precisely, we have

Theorem 1.2 (Continuous dependence)**.** *A family of weak solutions to the Cauchy problem* (1.1)∼(1.3) *can be constructed with the additional properties: For every* $t \in \mathbb{R}$ *we have*

$$\mathcal{E}(t) \;\le\; \mathcal{E}_0\,. \tag{1.26}$$

Moreover, let a sequence of initial conditions satisfy

$$\big\|(n_{i0}^k)_x - (n_{i0})_x\big\|_{L^2} \to 0, \qquad \big\|n_{i1}^k - n_{i1}\big\|_{L^2} \to 0\,, \quad i = 1 \sim 3,$$

and $\mathbf{n}_0^k \to \mathbf{n}_0$ *uniformly on compact sets, as* $k \to \infty$*. Then we have the convergence of the corresponding solutions* $\mathbf{n}^k \to \mathbf{n}$*, uniformly on bounded subsets of the* t-x *plane.*

It appears in (1.26) that the total energy of our solutions may decrease in time. Yet, we emphasize that our solutions are *conservative*, in the following sense.

Theorem 1.3 (Conservation of energy)**.** *There exists a continuous family* $\{\mu_t\,;\ t \in \mathbb{R}\}$ *of positive Radon measures on the real line with the following properties.*

- **(i):** *At every time* t*, we have* $\mu_t(\mathbb{R}) = \mathcal{E}_0$.
- **(ii):** *For each* t*, the absolutely continuous part of* μ_t *has density* $\frac{1}{2}\big(|\mathbf{n}_t|^2 + c^2(n_1)|\mathbf{n}_x|^2\big)$ *w.r.t. Lebesgue measure.*
- **(iii):** *For almost every* $t \in \mathbb{R}$*, the singular part of* μ_t *is concentrated on the set where* $n_1 = 0$ *or* ± 1*, when* $\alpha \ne \gamma$.

In other words, the total energy represented by the measure μ is conserved in time. Occasionally, some of this energy is concentrated on a set of measure zero. At those times τ when this happens, μ_τ has a non-trivial singular part and $\mathcal{E}(\tau) < \mathcal{E}_0$. Condition (iii) places some restrictions on the set of such times τ.

The paper is organized as follows. In Section 2 we introduce a new set of independent and dependent variables, and derive some identities valid for smooth solutions. We formulate a set of equations in the new variables which is equivalent to (1.1). Remarkably, in the new variables all singularities disappear: Smooth initial data lead to globally smooth solutions. In Section 3, we prove the existence of the solutions in the energy coordinates. In Sections 4∼7, we present proofs to Theorems 1.1∼1.3.

2. New formulation in energy-dependent coordinates

2.1. **Energy-dependent coordinates.** We denote

$$\vec{R}=(R_1,R_2,R_3)\stackrel{\text{def}}{=}\mathbf{n}_t+c(n_1)\mathbf{n}_x,\quad \vec{S}=(S_1,S_2,S_3)\stackrel{\text{def}}{=}\mathbf{n}_t-c(n_1)\mathbf{n}_x. \tag{2.1}$$

Then (1.1) can be reformulated as:

$$\begin{cases}\partial_t R_1-c(n_1)\partial_x R_1=\frac{1}{4c^2(n_1)}\big\{(c^2(n_1)-\gamma)(|\vec{R}|^2+|\vec{S}|^2)-2(3c^2(n_1)-\gamma)\vec{R}\cdot\vec{S}\big\}n_1\\ \qquad\qquad +\frac{c'(n_1)}{2c(n_1)}(R_1-S_1)R_1,\\ \partial_t S_1+c(n_1)\partial_x S_1=\frac{1}{4c^2(n_1)}\big\{(c^2(n_1)-\gamma)(|\vec{R}|^2+|\vec{S}|^2)-2(3c^2(n_1)-\gamma)\vec{R}\cdot\vec{S}\big\}n_1\\ \qquad\qquad -\frac{c'(n_1)}{2c(n_1)}(R_1-S_1)S_1,\\ \partial_t R_2-c(n_1)\partial_x R_2=\frac{1}{4c^2(n_1)}\big\{(c^2(n_1)-\alpha)(|\vec{R}|^2+|\vec{S}|^2)-2(3c^2(n_1)-\alpha)\vec{R}\cdot\vec{S}\big\}n_2\\ \qquad\qquad +\frac{c'(n_1)}{2c(n_1)}(R_2-S_2)R_1,\\ \partial_t S_2+c(n_1)\partial_x S_2=\frac{1}{4c^2(n_1)}\big\{(c^2(n_1)-\alpha)(|\vec{R}|^2+|\vec{S}|^2)-2(3c^2(n_1)-\alpha)\vec{R}\cdot\vec{S}\big\}n_2\\ \qquad\qquad -\frac{c'(n_1)}{2c(n_1)}(R_2-S_2)S_1,\\ \partial_t R_3-c(n_1)\partial_x R_3=\frac{1}{4c^2(n_1)}\big\{(c^2(n_1)-\alpha)(|\vec{R}|^2+|\vec{S}|^2)-2(3c^2(n_1)-\alpha)\vec{R}\cdot\vec{S}\big\}n_3\\ \qquad\qquad +\frac{c'(n_1)}{2c(n_1)}(R_3-S_3)R_1,\\ \partial_t S_3+c(n_1)\partial_x S_3=\frac{1}{4c^2(n_1)}\big\{(c^2(n_1)-\alpha)(|\vec{R}|^2+|\vec{S}|^2)-2(3c^2(n_1)-\alpha)\vec{R}\cdot\vec{S}\big\}n_3\\ \qquad\qquad -\frac{c'(n_1)}{2c(n_1)}(R_3-S_3)S_1,\\ \mathbf{n}_x=\frac{\vec{R}-\vec{S}}{2c(n_1)}\quad\text{or}\quad \mathbf{n}_t=\frac{\vec{R}+\vec{S}}{2}.\end{cases} \tag{2.2}$$

Similar to (1.14), system (2.2) has the following form of energy conservation law:

$$\frac{1}{4}\partial_t\big(|\vec{R}|^2+|\vec{S}|^2\big)-\frac{1}{4}\partial_x\big[c(n_1)(|\vec{R}|^2-|\vec{S}|^2)\big]=0. \tag{2.3}$$

We define the forward and backward characteristics as follows

$$\begin{cases}\frac{d}{ds}x^{\pm}(s,t,x)=\pm c(n_1(s,x^{\pm}(s,t,x))),\\ x^{\pm}|_{s=t}=x.\end{cases} \tag{2.4}$$

Then we define the coordinate transformation:

$$X\stackrel{\text{def}}{=}\int_0^{x^-(0,t,x)}[1+|\vec{R}|^2(0,y)]\,dy,\quad\text{and}\quad Y\stackrel{\text{def}}{=}\int_{x^+(0,t,x)}^0[1+|\vec{S}|^2(0,y)]\,dy.$$

This implies

$$X_t-c(n_1)X_x=0,\quad Y_t+c(n_1)Y_x=0. \tag{2.5}$$

Furthermore, for any smooth function f, we obtain by using (2.5) that

$$\begin{aligned}f_t+c(n_1)f_x&=(X_t+c(n_1)X_x)f_X=2c(n_1)X_xf_X\\ f_t-c(n_1)f_x&=(Y_t-c(n_1)Y_x)f_Y=-2c(n_1)Y_xf_Y.\end{aligned} \tag{2.6}$$

Introduce

$$p \overset{\text{def}}{=} \frac{1+|\vec{R}|^2}{X_x} \quad \text{and} \quad q \overset{\text{def}}{=} \frac{1+|\vec{S}|^2}{-Y_x}. \tag{2.7}$$

Then

$$\begin{aligned}\partial_t p - c(n_1)\partial_x p &= 2(X_x)^{-1}[\vec{R}\cdot(\partial_t\vec{R} - c(n_1)\partial_x\vec{R})] \\ &\quad -(X_x)^{-2}[\partial_t X_x - c(n_1)\partial_x X_x](1+|\vec{R}|^2).\end{aligned}$$

By (2.2), we have

$$\begin{aligned}\vec{R}\cdot(\partial_t\vec{R} - c(n_1)\partial_x\vec{R}) &= \frac{1}{4c^2(n_1)}\big\{(c^2(n_1)-\alpha)(|\vec{R}|^2+|\vec{S}|^2) - 2(3c^2(n_1)-\alpha)\vec{R}\cdot\vec{S}\big\}\vec{R}\cdot\mathbf{n} \\ &\quad + \frac{\alpha-\gamma}{4c^2(n_1)}\big(|\vec{R}|^2+|\vec{S}|^2 - 2\vec{R}\cdot\vec{S}\big)R_1 n_1 + \frac{c'(n_1)}{2c(n_1)}R_1(|\vec{R}|^2 - \vec{R}\cdot\vec{S}).\end{aligned}$$

Using $|\mathbf{n}| = 1$ and $c'(n_1) = \frac{(\gamma-\alpha)n_1}{c(n_1)}$, so that $\vec{R}\cdot\mathbf{n} = 0$, we further obtain

$$\vec{R}\cdot(\partial_t\vec{R} - c(n_1)\partial_x\vec{R}) = \frac{c'(n_1)}{4c(n_1)}R_1(|\vec{R}|^2 - |\vec{S}|^2), \tag{2.8}$$

which together with (2.5) applied gives

$$\partial_t p - c(n_1)\partial_x p = \frac{c'(n_1)}{2c(n_1)}\frac{p}{1+|\vec{R}|^2}[-R_1(1+|\vec{S}|^2) + S_1(1+|\vec{R}|^2)],$$

from which, and (2.6-2.7), we infer

$$\begin{aligned}p_Y &= \frac{1}{2c(n_1)(-Y_x)}(p_t - c(n_1)p_x) \\ &= \frac{c'(n_1)}{4c^2(n_1)}pq\Big(-\frac{R_1}{1+|\vec{R}|^2} + \frac{S_1}{1+|\vec{S}|^2}\Big).\end{aligned} \tag{2.9}$$

Similarly,

$$\begin{aligned}\partial_t q + c(n_1)\partial_x q &= 2(-Y_x)^{-1}[\vec{S}\cdot(\partial_t\vec{S} + c(n_1)\partial_x\vec{S})] \\ &\quad +(Y_x)^2(\partial_t Y_x + c(n_1)\partial_x Y_x)(1+|\vec{S}|^2),\end{aligned}$$

while again thanks to (2.2), we have

$$\begin{aligned}\vec{S}\cdot(\partial_t\vec{S} - c(n_1)\partial_x\vec{S}) &= \frac{1}{4c^2(n_1)}\big\{(c^2(n_1)-\alpha)(|\vec{R}|^2+|\vec{S}|^2) - 2(3c^2(n_1)-\alpha)\vec{R}\cdot\vec{S}\big\}\vec{S}\cdot\mathbf{n} \\ &\quad + \frac{\alpha-\gamma}{4c^2(n_1)}\big(|\vec{R}|^2+|\vec{S}|^2 - 2\vec{R}\cdot\vec{S}\big)S_1 n_1 + \frac{c'(n_1)}{2c(n_1)}S_1(|\vec{S}|^2 - \vec{R}\cdot\vec{S}),\end{aligned}$$

which along with the fact that $\vec{S}\cdot\mathbf{n} = 0$ leads to

$$\vec{S}\cdot(\partial_t\vec{S} - c(n_1)\partial_x\vec{S}) = -\frac{c'(n_1)}{4c(n_1)}S_1(|\vec{R}|^2 - |\vec{S}|^2), \tag{2.10}$$

from which and (2.5), we infer

$$\partial_t q + c(n_1)\partial_x q = \frac{c'(n_1)}{2c(n_1)}\frac{q}{1+|\vec{S}|^2}[R_1(1+|\vec{S}|^2) - S_1(1+|\vec{R}|^2)].$$

Then applying (2.6-2.7) gives rise to

$$
\begin{aligned}
q_X =&\frac{1}{2c(n_1)X_x}(q_t + c(n_1)q_x)\\
=&\frac{c'(n_1)}{4c^2(n_1)}pq\Big(\frac{R_1}{1+|\vec{R}|^2}-\frac{S_1}{1+|\vec{S}|^2}\Big).
\end{aligned}
\tag{2.11}
$$

Now we introduce

$$
\begin{aligned}
&\vec{\ell}=(\ell_1,\ell_2,\ell_3)\overset{\text{def}}{=}\frac{\vec{R}}{1+|\vec{R}|^2},\quad \vec{m}=(m_1,m_2,m_3)\overset{\text{def}}{=}\frac{\vec{S}}{1+|\vec{S}|^2},\quad\text{and}\\
&h_1\overset{\text{def}}{=}\frac{1}{1+|\vec{R}|^2},\quad h_2\overset{\text{def}}{=}\frac{1}{1+|\vec{S}|^2}.
\end{aligned}
\tag{2.12}
$$

Then

$$
\begin{aligned}
\partial_t\ell_1 - c(n_1)\partial_x\ell_1 \;=\;& (1+|\vec{R}|^2)^{-1}(\partial_t R_1 - c(n_1)\partial_x R_1)\\
&-2(1+|\vec{R}|^2)^{-2}R_1\big[\vec{R}\cdot(\partial_t\vec{R}-c(n_1)\partial_x\vec{R})\big],
\end{aligned}
$$

which together with (2.2) and (2.8) imply

$$
\begin{aligned}
\partial_t\ell_1 - c(n_1)\partial_x\ell_1 =&(1+|\vec{R}|^2)^{-1}\Big[\frac{c^2(n_1)-\gamma}{4c^2(n_1)}(|\vec{R}|^2+|\vec{S}|^2)-\frac{3c^2(n_1)-\gamma}{2c^2(n_1)}\vec{R}\cdot\vec{S}\Big]n_1\\
&+\frac{c'(n_1)}{2c(n_1)}(1+|\vec{R}|^2)^{-2}R_1\big[R_1(1+|\vec{S}|^2)-S_1(1+|\vec{R}|^2)\big].
\end{aligned}
$$

Then thanks to (2.6-2.7), we obtain

$$
\begin{aligned}
\partial_Y\ell_1 =&\frac{q}{8c^3(n_1)}\big[(c^2(n_1)-\gamma)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\gamma)\vec{\ell}\cdot\vec{m}\big]n_1\\
&+\frac{c'(n_1)}{4c^2(n_1)}\ell_1 q(\ell_1-m_1).
\end{aligned}
\tag{2.13}
$$

Similarly thanks to (2.12), we have

$$
\begin{aligned}
\partial_t m_1 + c(n_1)\partial_x m_1 =&(1+|\vec{S}|^2)^{-1}(\partial_t S_1 + c(n_1)\partial_x S_1)\\
&-2(1+|\vec{S}|^2)^{-2}S_1\big[\vec{S}\cdot(\partial_t\vec{S}+c(n_1)\partial_x\vec{S})\big],
\end{aligned}
$$

which together with (2.2) and (2.10) ensure that

$$
\begin{aligned}
\partial_t m_1 + c(n_1)\partial_x m_1 =&(1+|\vec{S}|^2)^{-1}\Big[\frac{c^2(n_1)-\gamma}{4c^2(n_1)}(|\vec{R}|^2+|\vec{S}|^2)-\frac{3c^2(n_1)-\gamma}{2c^2(n_1)}\vec{R}\cdot\vec{S}\Big]n_1\\
&-\frac{c'(n_1)}{2c(n_1)}(1+|\vec{S}|^2)^{-2}S_1\big[R_1(1+|\vec{S}|^2)-S_1(1+|\vec{R}|^2)\big],
\end{aligned}
$$

from which and (2.6-2.7), we infer

$$
\begin{aligned}
\partial_X m_1 =&\frac{p}{8c^3(n_1)}\big[(c^2(n_1)-\gamma)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\gamma)\vec{\ell}\cdot\vec{m}\big]n_1\\
&-\frac{c'(n_1)}{4c^2(n_1)}m_1 p(\ell_1-m_1).
\end{aligned}
\tag{2.14}
$$

Following the same line, we deduce from (2.2) and (2.8) that

$$\partial_t \ell_2 - c(n_1)\partial_x \ell_2 = (1+|\vec{R}|^2)^{-1}\Big[\frac{c^2(n_1)-\alpha}{4c^2(n_1)}(|\vec{R}|^2+|\vec{S}|^2) - \frac{3c^2(n_1)-\alpha}{2c^2(n_1)}\vec{R}\cdot\vec{S}\Big]n_2 + \frac{c'(n_1)}{2c(n_1)}(1+|\vec{R}|^2)^{-2}R_1\big[R_2(1+|\vec{S}|^2) - S_2(1+|\vec{R}|^2)\big],$$

and

$$\partial_t \ell_3 - c(n_1)\partial_x \ell_3 = (1+|\vec{R}|^2)^{-1}\Big[\frac{c^2(n_1)-\alpha}{4c^2(n_1)}(|\vec{R}|^2+|\vec{S}|^2) - \frac{3c^2(n_1)-\alpha}{2c^2(n_1)}\vec{R}\cdot\vec{S}\Big]n_3 + \frac{c'(n_1)}{2c(n_1)}(1+|\vec{R}|^2)^{-2}R_1\big[R_3(1+|\vec{S}|^2) - S_3(1+|\vec{R}|^2)\big],$$

which together with (2.6-2.7) ensure that

$$\begin{aligned}\partial_Y \ell_2 =& \frac{q}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]n_2\\ &+ \frac{c'(n_1)}{4c^2(n_1)}\ell_1 q(\ell_2 - m_2) \qquad \text{and}\\ \partial_Y \ell_3 =& \frac{q}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]n_3\\ &+ \frac{c'(n_1)}{4c^2(n_1)}\ell_1 q(\ell_3 - m_3).\end{aligned} \tag{2.15}$$

While we deduce from (2.2) and (2.10) that

$$\partial_t m_2 + c(n_1)\partial_x m_2 = (1+|\vec{S}|^2)^{-1}\Big[\frac{c^2(n_1)-\alpha}{4c^2(n_1)}(|\vec{R}|^2+|\vec{S}|^2) - \frac{3c^2(n_1)-\alpha}{2c^2(n_1)}\vec{R}\cdot\vec{S}\Big]n_2 - \frac{c'(n_1)}{2c(n_1)}(1+|\vec{S}|^2)^{-2}S_1\big[R_2(1+|\vec{S}|^2) - S_2(1+|\vec{R}|^2)\big],$$

and

$$\partial_t m_3 + c(n_1)\partial_x m_3 = (1+|\vec{S}|^2)^{-1}\Big[\frac{c^2(n_1)-\alpha}{4c^2(n_1)}(|\vec{R}|^2+|\vec{S}|^2) - \frac{3c^2(n_1)-\alpha}{2c^2(n_1)}\vec{R}\cdot\vec{S}\Big]n_3 - \frac{c'(n_1)}{2c(n_1)}(1+|\vec{S}|^2)^{-2}S_1\big[R_3(1+|\vec{S}|^2) - S_3(1+|\vec{R}|^2)\big],$$

which together with (2.6-2.7) imply that

$$\begin{aligned}\partial_X m_2 =& \frac{p}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]n_2\\ &- \frac{c'(n_1)}{4c^2(n_1)}m_1 p(\ell_2 - m_2) \qquad \text{and}\\ \partial_X m_3 =& \frac{p}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]n_3\\ &- \frac{c'(n_1)}{4c^2(n_1)}m_1 p(\ell_3 - m_3).\end{aligned} \tag{2.16}$$

On the other hand, it follows from (2.8) and (2.12) that

$$\begin{aligned}\partial_t h_1 - c(n_1)\partial_x h_1 =& -2(1+|\vec{R}|^2)^{-2}\big[\vec{R}\cdot(\partial_t\vec{R}-c(n_1)\partial_x\vec{R})\big]\\ =& -\frac{c'(n_1)}{2c(n_1)}(1+|\vec{R}|^2)^{-2}R_1(|\vec{R}|^2-|\vec{S}|^2).\end{aligned}$$

Then we obtain by using (2.6-2.7) that

$$\partial_Y h_1 = \frac{c'(n_1)}{4c^2(n_1)}q\ell_1(h_1-h_2). \tag{2.17}$$

Similar calculations together with (2.10) yield

$$\partial_t h_2 + c(n_1)\partial_x h_2 = \frac{c'(n_1)}{2c(n_1)}(1+|\vec{S}|^2)^{-2}S_1(|\vec{R}|^2-|\vec{S}|^2),$$

which together with (2.6-2.7) imply that

$$\partial_X h_2 = \frac{c'(n_1)}{4c^2(n_1)}pm_1(h_2-h_1). \tag{2.18}$$

Finally we observe from (2.6) and (2.12) that

$$\begin{aligned}\partial_Y\mathbf{n} =& \frac{1}{2c(n_1)X_x}(\partial_t\mathbf{n}-c(n_1)\partial_x\mathbf{n}) = \frac{q}{2c(n_1)}\vec{m},\\ \partial_X\mathbf{n} =& \frac{1}{2c(n_1)(-Y_x)}(\partial_t\mathbf{n}+c(n_1)\partial_x\mathbf{n}) = \frac{p}{2c(n_1)}\vec{\ell}.\end{aligned} \tag{2.19}$$

In summary, we obtain

$$\begin{cases}\partial_Y\ell_1 = \frac{q}{8c^3(n_1)}\big[(c^2(n_1)-\gamma)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\gamma)\vec{\ell}\cdot\vec{m}\big]n_1\\ \qquad +\frac{c'(n_1)}{4c^2(n_1)}\ell_1 q(\ell_1-m_1),\\ \partial_X m_1 = \frac{p}{8c^3(n_1)}\big[(c^2(n_1)-\gamma)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\gamma)\vec{\ell}\cdot\vec{m}\big]n_1\\ \qquad -\frac{c'(n_1)}{4c^2(n_1)}m_1 p(\ell_1-m_1),\\ \partial_Y\ell_2 = \frac{q}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]n_2\\ \qquad +\frac{c'(n_1)}{4c^2(n_1)}\ell_1 q(\ell_2-m_2),\\ \partial_X m_2 = \frac{p}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]n_2\\ \qquad -\frac{c'(n_1)}{4c^2(n_1)}m_1 p(\ell_2-m_2),\\ \partial_Y\ell_3 = \frac{q}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]n_3\\ \qquad +\frac{c'(n_1)}{4c^2(n_1)}\ell_1 q(\ell_3-m_3),\\ \partial_X m_3 = \frac{p}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]n_3\\ \qquad -\frac{c'(n_1)}{4c^2(n_1)}m_1 p(\ell_3-m_3),\\ \partial_Y\mathbf{n} = \frac{q}{2c(n_1)}\vec{m}, \quad (\text{or} \quad \partial_X\mathbf{n} = \frac{p}{2c(n_1)}\vec{\ell})\\ \partial_Y h_1 = \frac{c'(n_1)}{4c^2(n_1)}q\ell_1(h_1-h_2), \qquad \partial_X h_2 = \frac{c'(n_1)}{4c^2(n_1)}m_1 p(h_2-h_1),\\ p_Y = -\frac{c'(n_1)}{4c^2(n_1)}pq(\ell_1-m_1), \qquad q_X = \frac{c'(n_1)}{4c^2(n_1)}pq(\ell_1-m_1).\end{cases} \tag{2.20}$$

2.2. **Consistency of variables of** (2.20)**.** The various variables introduced for (2.20) are consistent, following from the proposition below.

Proposition 2.1. *For smooth enough data, there hold the following conservative quantities:*

$$\begin{aligned}&\vec{\ell}\cdot\mathbf{n}(X,Y)=\vec{m}\cdot\mathbf{n}(X,Y)=0,\quad |\mathbf{n}(X,Y)|=1\quad\textit{and}\\&|\vec{\ell}(X,Y)|^2+h_1^2(X,Y)=h_1(X,Y),\quad |\vec{m}(X,Y)|^2+h_2^2(X,Y)=h_2(X,Y),\quad\forall\ X,Y,\end{aligned}\tag{2.21}$$

as long as

$$\begin{aligned}&\vec{\ell}\cdot\mathbf{n}(X,\varphi(X))=\vec{m}\cdot\mathbf{n}(X,\varphi(X))=0,\quad |\mathbf{n}(X,\varphi(X))|=1\quad\textit{and}\\&|\vec{\ell}(X,\varphi(X))|^2+h_1^2(X,\varphi(X))=h_1(X,\varphi(X)),\\&|\vec{m}(X,\varphi(X))|^2+h_2^2(X,\varphi(X))=h_2(X,\varphi(X)),\quad\forall\ X.\end{aligned}$$

Here the function φ represents the initial curve, see (3.2) *from the next section.*

Proof. We first deduce from (2.20) that

$$\begin{aligned}\partial_Y\big[|\vec{\ell}|^2+h_1^2-h_1\big]=&\frac{q}{4c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]\vec{\ell}\cdot\mathbf{n}\\&+\frac{(\alpha-\gamma)n_1}{4c^3(n_1)}q\big[(h_1+h_2-2h_1h_2)-2\vec{\ell}\cdot\vec{m}\big]\ell_1\\&+\frac{c'(n_1)}{4c^2(n_1)}q\ell_1\big[2(|\vec{\ell}|^2-\vec{\ell}\cdot\vec{m})+2h_1(h_1-h_2)-(h_1-h_2)\big],\end{aligned}$$

from which and the fact that $c'(n_1)=\frac{(\gamma-\alpha)n_1}{c(n_1)}$, we deduce that

$$\begin{aligned}\partial_Y\big[|\vec{\ell}|^2+h_1^2-h_1\big]=&\frac{q}{4c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2)\\&-2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]\vec{\ell}\cdot\mathbf{n}+\frac{c'(n_1)}{2c^2(n_1)}q\ell_1\big[|\vec{\ell}|^2+h_1^2-h_1\big].\end{aligned}\tag{2.22}$$

Similarly it follows from (2.20) that

$$\begin{aligned}\partial_Y[\vec{\ell}\cdot\mathbf{n}]=&\partial_Y\vec{\ell}\cdot\mathbf{n}+\vec{\ell}\cdot\partial_Y\mathbf{n}\\=&\frac{q}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big](|\mathbf{n}|^2-1)\\&+\frac{q}{8c^3(n_1)}(h_1+h_2-2h_1h_2)\big[(c^2(n_1)-\alpha)+(\alpha-\gamma)n_1^2\big]\\&-\frac{q}{4c^3(n_1)}\vec{\ell}\cdot\vec{m}\big[3c^2(n_1)-\alpha+(\alpha-\gamma)n_1^2\big]\\&+\frac{q}{4c^2(n_1)}q\ell_1[\vec{\ell}\cdot\mathbf{n}-\vec{m}\cdot\mathbf{n}]+\frac{q}{2c(n_1)}\vec{\ell}\cdot\vec{m},\end{aligned}$$

which along with the fact that $c^2(n_1)=\alpha+(\gamma-\alpha)n_1^2$ leads to

$$\begin{aligned}\partial_Y[\vec{\ell}\cdot\mathbf{n}]=&\frac{q}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2)-2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big](|\mathbf{n}|^2-1)\\&+\frac{c'(n_1)}{4c^2(n_1)}q\ell_1[\vec{\ell}\cdot\mathbf{n}-\vec{m}\cdot\mathbf{n}].\end{aligned}\tag{2.23}$$

On the other hand, observe that $\partial_Y \mathbf{n} = \frac{q}{2c(n_1)}\vec{m}$ is consistent with $\partial_X \mathbf{n} = \frac{p}{2c(n_1)}\vec{\ell}$. Indeed again thanks to (2.20), one has on the one hand

$$\partial_X[\partial_Y \mathbf{n}] = -\frac{c'(n_1)}{8c^3(n_1)}pq(\ell_1+m_1)\vec{m} + \frac{q}{2c(n_1)}\partial_X\vec{m},$$

and on the other hand

$$\partial_Y[\partial_X \mathbf{n}] = -\frac{c'(n_1)}{8c^3(n_1)}pq(\ell_1+m_1)\vec{\ell} + \frac{p}{2c(n_1)}\partial_Y\vec{\ell},$$

which along with the $\vec{\ell}$ equations and $\vec{m}$ equations of (2.20) shows that $\partial_X[\partial_Y \mathbf{n}] = \partial_Y[\partial_X \mathbf{n}]$. So we can also use the equation $\partial_X \mathbf{n} = \frac{p}{2c(n_1)}\vec{\ell}$, from which and the $\vec{m}$ equation of (2.20), we obtain

$$\begin{aligned}\partial_X[\vec{m}\cdot\mathbf{n}] =& \partial_X\vec{m}\cdot\mathbf{n} + \vec{m}\cdot\partial_X\vec{\ell}\\
=& \frac{p}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big](|\mathbf{n}|^2-1)\\
&+ \frac{p}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]\\
&+ \frac{\alpha-\gamma}{8c^3(n_1)}pn_1^2\big[(h_1+h_2-2h_1h_2) - 2\vec{\ell}\cdot\vec{m}\big]\\
&+ \frac{c'(n_1)}{4c^2(n_1)}pm_1[\vec{m}\cdot\mathbf{n} - \vec{\ell}\cdot\mathbf{n}] + \frac{p}{2c(n_1)}\vec{\ell}\cdot\vec{m},\end{aligned}$$

which gives rise to

$$\begin{aligned}\partial_Y[\vec{m}\cdot\mathbf{n}] =& \frac{p}{8c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big](|\mathbf{n}|^2-1)\\
&- \frac{c'(n_1)}{4c^2(n_1)}pm_1[\vec{\ell}\cdot\mathbf{n} - \vec{m}\cdot\mathbf{n}].\end{aligned} \tag{2.24}$$

Next it is easy to observe that

$$\partial_Y[|\mathbf{n}|^2-1] = \frac{q}{2c(n_1)}\vec{m}\cdot\mathbf{n}. \tag{2.25}$$

Finally, to control the evolution of $|\vec{m}|^2 + h_2^2 - h_2$, we obtain by applying (2.20) that

$$\begin{aligned}\partial_X\big[|\vec{m}|^2 + h_2^2 - h_2\big] =& \frac{p}{4c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]\vec{m}\cdot\mathbf{n}\\
&+ \frac{\alpha-\gamma}{4c^3(n_1)}p\big[(h_1+h_2-2h_1h_2) - 2\vec{\ell}\cdot\vec{m}\big]m_1n_1\\
&+ \frac{c'(n_1)}{2c^2(n_1)}pm_1\big[(|\vec{m}|^2 - \vec{\ell}\cdot\vec{m}) + (h_2-\frac{1}{2})(h_2-h_1)\big],\end{aligned}$$

which simplifies to

$$\begin{aligned}\partial_X\big[|\vec{m}|^2 + h_2^2 - h_2\big] =& \frac{p}{4c^3(n_1)}\big[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2)\\
&- 2(3c^2(n_1)-\alpha)\vec{\ell}\cdot\vec{m}\big]\vec{m}\cdot\mathbf{n} + \frac{c'(n_1)}{2c^3(n_1)}pm_1\big[|\vec{m}|^2 + h_2^2 - h_2\big].\end{aligned} \tag{2.26}$$

Summing up (2.22) to (2.26) gives rise to (2.21). This completes the proof of the proposition. □

3. Solutions in the energy coordinates

Using (2.6) by letting $f = t$ or x, we obtain the equations

$$t_X = \frac{ph_1}{2c}, \quad t_Y = \frac{qh_2}{2c}, \quad x_X = \frac{ph_1}{2}, \quad x_Y = \frac{-qh_2}{2}. \tag{3.1}$$

Only one of the two equations of t_X and t_Y is needed for recovering t, and the two equations are consistent since $t_{XY} = t_{YX}$. The same is true for x.

The initial line $t = 0$ in the (x,t) plane is transformed to a parametric curve

$$\gamma : \quad Y = \varphi(X) \tag{3.2}$$

in the (X,Y) plane, where $Y = \varphi(X)$ if and only if there is an x such that

$$\begin{cases} X &= \int_0^x [1 + |\vec{R}|^2(0,y)]\, dy, \\ Y &= \int_x^0 [1 + |\vec{S}|^2(0,y)]\, dy. \end{cases} \tag{3.3}$$

We point out that the curve is non-characteristic. We introduce

$$\mathcal{E}_0 \ := \ \frac{1}{4}\int \Big[|\vec{R}|^2(0,y) + |\vec{S}|^2(0,y)\Big]\, dy < \infty. \tag{3.4}$$

It equals to the number in (1.25). The two functions $X = X(x), Y = Y(x)$ from (3.3) are well-defined and absolutely continuous, provided that (1.2) is satisfied. Clearly, X is strictly increasing while Y is strictly decreasing. Therefore, the map $X \mapsto \varphi(X)$ is continuous and strictly decreasing. From (3.4) it follows

$$\big|X + \varphi(X)\big| \le 4\mathcal{E}_0\,. \tag{3.5}$$

As (t,x) ranges over the domain $[0,\infty) \times \mathbb{R}$, the corresponding variables (X,Y) range over the set

$$\Omega^+ := \big\{(X,Y)\,;\ Y \ge \varphi(X)\big\}\,. \tag{3.6}$$

Along the curve

$$\gamma := \big\{(X,Y)\,;\ Y = \varphi(X)\big\} \subset \mathbb{R}^2$$

parametrized by $x \mapsto \big(X(x),\, Y(x)\big)$, we can thus assign the boundary data $(\bar{\vec{\ell}}, \bar{\vec{m}}, \bar{h}_1, \bar{h}_2, \bar{p}, \bar{q}, \bar{\mathbf{n}})$ $\in L^\infty$ defined by their definition evaluated at the initial data (1.2), i.e.,

$$\begin{aligned} &\bar{\mathbf{n}} = \mathbf{n}_0(x)\,, \quad \bar{p} = 1\,, \quad \bar{q} = 1\,, \\ &\bar{\vec{\ell}} = \vec{R}(0,x)\bar{h}_1, \quad \bar{\vec{m}} = \vec{S}(0,x)\bar{h}_2, \\ &\bar{h}_1 = \frac{1}{1 + |\vec{R}|^2(0,x)}, \qquad \text{and} \\ &\bar{h}_2 = \frac{1}{1 + |\vec{S}|^2(0,x)}. \end{aligned} \tag{3.7}$$

where

$$\vec{R}(0,x) = \mathbf{n}_1(x) + c(n_{10}(x))\mathbf{n}_0'(x), \quad \vec{S}(0,x) = \mathbf{n}_1(x) - c(n_{10}(x))\mathbf{n}_0'(x).$$

We consider solutions to the boundary value problem (2.20)(3.7)(1.3).

Theorem 3.1. *The problem (2.20)(3.7)(1.2)(1.3) has a unique global solution defined for all* $(X,Y) \in \mathbb{R}^2$.

Sketch of Proof. In the following, we shall construct the solution on the domain Ω^+ where $Y \geq \varphi(X)$. On the complementary set Ω^- where $Y < \varphi(X)$, the solution can be constructed similarly.

Observing that all equations in (2.20) have a locally Lipschitz continuous right hand side, the construction of a local solution as fixed point of a suitable integral transformation is straightforward. To make sure that this solution is actually defined on the whole domain Ω^+, one must establish *a priori* bounds, showing that the solution remains bounded on bounded sets.

By (2.21), we have

$$h_1(1-h_1) \geq 0, \quad h_2(1-h_2) \geq 0. \tag{3.8}$$

Thus h_1, h_2 are bounded between zero and one, and both $|\vec{\ell}|$ and $|\vec{m}|$ are uniformly bounded.

By the p and q equations in (2.20), we have

$$p_Y + q_X = 0 \tag{3.9}$$

which implies that

$$\int_{\varphi^{-1}(Y)}^{X} p(X', Y) dX' + \int_{\varphi(X)}^{Y} q(X, Y') dY' = X - \varphi^{-1}(Y) + Y - \varphi(X)$$

where φ^{-1} denotes the inverse of φ, following an integration over the characteristic triangle with vertex (X, Y). Thus, by the energy assumption (3.4), we find

$$\int_{\varphi^{-1}(Y)}^{X} p(X', Y) dX' + \int_{\varphi(X)}^{Y} q(X, Y')\, dY' \leq 2(|X| + |Y| + 4\mathcal{E}_0). \tag{3.10}$$

Integrating the p equation in (2.20) vertically and use the bound on q from (3.10), we find

$$\begin{aligned} p(X,Y) &= \exp\left\{ \int_{\varphi(X)}^{Y} \tfrac{c'(n_1)}{4c^2(n_1)} q(X,Y')(-\ell_1 + m_1)\, dY' \right\} \\ &\leq \exp\left\{ C_0 \int_{\varphi(X)}^{Y} q(X,Y')\, dY' \right\} \\ &\leq \exp\{2C_0(|X| + |Y| + 4\mathcal{E}_0)\}. \end{aligned} \tag{3.11}$$

Here C_0 represents a finite number. Similarly, we have

$$q(X,Y) \leq \exp\{2C_0(|X| + |Y| + 4\mathcal{E}_0)\}. \tag{3.12}$$

Relying on the local bounds (3.11)(3.12), the local solution to (2.20)(3.7)(1.2)(1.3) can be extended to the entire plane. One may consult paper [3] for details. This completes the sketch of the proof.

Let us state a useful consequence of the above construction for future reference.

Corollary 3.1. *If the initial data $(\mathbf{n}_0, \mathbf{n}_1)$ are smooth, the solution $U := (\mathbf{n}, p, q, \vec{\ell}, \vec{m}, h_1, h_2)$ of (2.20)(3.7)(1.2)(1.3) is a smooth function of the variables (X, Y). Moreover, assume that a sequence of smooth functions $(\mathbf{n}_0^i, \mathbf{n}_1^i)_{i\geq 1}$ satisfies*

$$\mathbf{n}_0^i \to \mathbf{n}_0\,, \qquad (\mathbf{n}_0^i)_x \to (\mathbf{n}_0)_x\,, \qquad \mathbf{n}_1^i \to \mathbf{n}_1$$

uniformly on compact subsets of $\mathbb{R}$. Then one has the convergence of the corresponding solutions:

$$(\mathbf{n}^i, p^i, q^i, \vec{\ell}^i, \vec{m}^i, h_1^i, h_2^i) \to U$$

uniformly on bounded subsets of the X-Y plane.

4. Inverse transformation

By expressing the solution $\mathbf{n}(X,Y)$ in terms of the original variables (t,x), we shall recover a solution of the Cauchy problem (1.1)∼(1.3). This will provide a proof of Theorem 1.1.

We integrate (3.1) with data $t=0, x=x$ on γ to find $t=t(X,Y), x=x(X,Y)$, which exist for all (X,Y) in $\mathbb{R}^2$.

We need the inverse functions $X=X(t,x), Y=Y(t,x)$. The inverse functions do not exist as a one-to-one correspondence between (t,x) in $\mathbb{R}^2$ and (X,Y) in $\mathbb{R}^2$. There may be a nontrivial set of points in the (X,Y) plane that maps to a single point (t,x). To investigate it, we find the partial derivatives of the inverse mapping, valid at points where $h_1 \neq 0, h_2 \neq 0$,

$$X_t = \frac{c}{ph_1}, \quad Y_t = \frac{c}{qh_2}, \quad X_x = \frac{1}{ph_1}, \quad Y_x = -\frac{1}{qh_2}. \tag{4.1}$$

Thus (2.5) holds and so does (2.6) for our solution.

As a preliminary, we examine the regularity of the solution $U = (\mathbf{n}, p, q, \vec{\ell}, ...)$ constructed in the previous section. Since the initial data $(\mathbf{n}_0)_x, \mathbf{n}_1$ etc. are only assumed to be in L^2, the functions U may well be discontinuous. More precisely, on bounded subsets of the X-Y plane, the solutions satisfy the following:

- **-:** The functions $\ell_1, \ell_2, , \ell_3, h_1, p$ are Lipschitz continuous w.r.t. Y, measurable w.r.t. X.
- **-:** The functions m_1, m_2, m_3, h_2, q are Lipschitz continuous w.r.t. X, measurable w.r.t. Y.
- **-:** The vector field $\mathbf{n}$ is Lipschitz continuous w.r.t. both X and Y.

In order to define $\mathbf{n}$ as a vector field of the original variables t, x, we should formally invert the map $(X,Y) \mapsto (t,x)$ and write $\mathbf{n}(t,x) = \mathbf{n}\big(X(t,x), Y(t,x)\big)$. The fact that the above map may not be one-to-one does not cause any real difficulty. Indeed, given (t^*, x^*), we can choose an arbitrary point (X^*, Y^*) such that $t(X^*,Y^*) = t^*$, $x(X^*,Y^*) = x^*$, and define $\mathbf{n}(t^*,x^*) = \mathbf{n}(X^*,Y^*)$. To prove that the values of $\mathbf{n}$ do not depend on the choice of (X^*,Y^*), we proceed as follows. Assume that there are two distinct points such that $t(X_1,Y_1) = t(X_2,Y_2) = t^*$, $x(X_1,Y_1) = x(X_2,Y_2) = x^*$. We consider two cases:
Case 1: $X_1 \le X_2$, $Y_1 \le Y_2$. Consider the set

$$\Gamma_{x^*} := \Big\{(X,Y)\,;\ x(X,Y) \le x^*\Big\}$$

and call $\partial\Gamma_{x^*}$ its boundary. By (3.1), x is increasing in X and decreasing in Y. Hence, this boundary can be represented as the graph of a Lipschitz continuous function: $X - Y = \phi(X+Y)$. We now construct the Lipschitz continuous curve γ consisting of

- **-:** a horizontal segment joining (X_1,Y_1) with a point $A = (X_A, Y_A)$ on $\partial\Gamma_{x^*}$, with $Y_A = Y_1$,
- **-:** a portion of the boundary $\partial\Gamma_{x^*}$,
- **-:** a vertical segment joining (X_2,Y_2) to a point $B = (X_B,Y_B)$ on $\partial\Gamma_{x^*}$, with $X_B = X_2$.

Observe that the map $(X,Y) \mapsto (t,x)$ is constant along γ. By (3.1) this implies $h_1 = 0$ on the horizontal segment, $h_2 = 0$ on the vertical segment, and $h_1 = h_2 = 0$ on the portion of the boundary $\partial\Gamma_{x^*}$. When either $h_1 = 0$ or $h_2 = 0$ or both, we have from the conserved quantities (2.22) that $\vec{\ell} = 0$ or $\vec{m} = 0$ or both, correspondingly. Upon examining the derivatives of $\mathbf{n}$ in (2.20), we have the same pattern of vanishing property. Thus, along the path from A to B, the values of the components of $\mathbf{n}$ remain constant, proving our claim.
Case 2: $X_1 \le X_2$, $Y_1 \ge Y_2$. In this case, we consider the set

$$\Gamma_{t^*} := \Big\{(X,Y)\,;\ t(X,Y) \le t^*\Big\},$$

and construct a curve γ connecting (X_1, Y_1) with (X_2, Y_2) similarly as in case 1. Details are entirely similar to Case 1.

We now prove that the function $\mathbf{n}(t,x) = \mathbf{n}\big(X(t,x), Y(t,x)\big)$ thus obtained are Hölder continuous on bounded sets. Toward this goal, consider any characteristic curve, say $t \mapsto x^+(t)$, with $dx^+/dt = c(n_1)$. By construction, this is parametrized by the function $X \mapsto \big(t(X,\overline{Y}),\, x(X,\overline{Y})\big)$, for some fixed $\overline{Y}$. Using the chain rule and the inverse mapping formulas (4.1), we obtain

$$\mathbf{n}_t + c\mathbf{n}_x = \mathbf{n}_X(X_t + cX_x) + \mathbf{n}_Y(Y_t + cY_x) = 2cX_x\mathbf{n}_X.$$

Thus we have

$$\begin{aligned}
\int_0^\tau \big|\mathbf{n}_t + c\mathbf{n}_x\big|^2 dt &= \int_{X_0}^{X_\tau} (2cX_x|\mathbf{n}_X|)^2\,(2X_t)^{-1} dX \\
&= \int_{X_0}^{X_\tau} (2c\tfrac{1}{ph_1}\tfrac{p|\vec{\ell}|}{2c})^2\tfrac{ph_1}{2c}\,dX \\
&= \int_{X_0}^{X_\tau} \tfrac{p}{2c}\tfrac{|\vec{\ell}|^2}{h_1}\,dX \;\le\; \int_{X_0}^{X_\tau}\tfrac{p}{2c}\,dX \;\le\; C_\tau\,,
\end{aligned} \tag{4.2}$$

for some constant C_τ depending only on τ. Notice we have used $|\vec{\ell}|^2 \le h_1$, which follows from (2.21). Similarly, integrating along any backward characteristics $t \mapsto x^-(t)$ we obtain

$$\int_0^\tau \big|\mathbf{n}_t - c\mathbf{n}_x\big|^2 dt \le C_\tau. \tag{4.3}$$

Since the speed of characteristics is $\pm c(n_1)$, and $c(n_1)$ is uniformly positive and bounded, the bounds (4.2)-(4.3) imply that the function $\mathbf{n} = \mathbf{n}(t,x)$ is Hölder continuous with exponent 1/2. In turn, this implies that all characteristic curves are $\mathbf{C}^1$ with Hölder continuous derivative. In addition, from (4.2)-(4.3) it follows that $\vec{R}, \vec{S}$ at (2.1) are square integrable on bounded subsets of the t-x plane. However, we should check the consistency that $\vec{R}, \vec{S}$ at (2.1) are indeed the same as recovered from (2.12). Let us check only one of them, $R = \vec{\ell}/h_1$. We find

$$\mathbf{n}_t + c\mathbf{n}_x = 2cX_x\mathbf{n}_X = 2c\frac{1}{ph_1}\frac{p\vec{\ell}}{2c} = \frac{\vec{\ell}}{h_1}.$$

Finally, we prove that $\mathbf{n}$ satisfies the equations of system (1.1) in distributional sense, according to (iii) of Definition 1.1. We note that

$$\begin{aligned}
&\iint 2\big[\phi_t\mathbf{n}_t - \phi_x c^2\mathbf{n}_x\big]\,dxdt \\
&= \iint \phi_t\big[(\mathbf{n}_t + c\mathbf{n}_x) + (\mathbf{n}_t - c\mathbf{n}_x)\big] \\
&\qquad -c\phi_x\big[(\mathbf{n}_t + c\mathbf{n}_x) - (\mathbf{n}_t - c\mathbf{n}_x)\big]\,dxdt \\
&= \iint \big[\phi_t - c\phi_x\big]\,(\mathbf{n}_t + c\mathbf{n}_x)\,dxdt \\
&\quad + \iint \big[\phi_t + c\phi_x\big]\,(\mathbf{n}_t - c\mathbf{n}_x)\,dxdt \\
&= \iint \big[\phi_t - c\phi_x\big]\,\vec{R}\,dxdt + \iint \big[\phi_t + c\phi_x\big]\,\vec{S}\,dxdt\,.
\end{aligned} \tag{4.4}$$

By (2.6), this is equal to

$$\iint \big[-2cY_x\phi_Y\,\vec{R} \;+\; 2cX_x\phi_X\,\vec{S}\big]\,dxdt\,. \tag{4.5}$$

Using the Jacobian

$$\frac{\partial(x,t)}{\partial(X,Y)} = \frac{pqh_1h_2}{2c} \tag{4.6}$$

derived from (3.1), and the inverse (4.1), we see that it is equal to

$$
\begin{aligned}
&\iint \Big[\frac{2c}{qh_2}\vec{R}\phi_Y + \frac{2c}{ph_1}\vec{S}\phi_X\Big]\frac{pqh_1h_2}{2c}\,dXdY\\
=&\iint \big[ph_1\vec{R}\phi_Y + qh_2\vec{S}\phi_X\big]\,dXdY\\
=&\iint \big[p\vec{\ell}\phi_Y + q\vec{m}\phi_X\big]\,dXdY\\
=&\iint \big[-(p\vec{\ell})_Y - (q\vec{m})_X\big]\phi\,dXdY.
\end{aligned}
\tag{4.7}
$$

Thanks to (2.1), (2.12) and (2.20), we have that the first component of the integrand equals to

$$
\begin{aligned}
&\phi\big[-(p\ell_1)_Y - (qm_1)_X\big]\\
=&-\phi\frac{pq}{4c^3}\big[(c^2-\gamma)(h_1+h_2-2h_1h_2) - 2(3c^2-\gamma)\vec{\ell}\cdot\vec{m}\big]n_1\\
=&-\phi\frac{pqh_1h_2}{2c}\big[\frac{c^2-\gamma}{2c^2}(\frac{1}{h_2}+\frac{1}{h_1}-2) - \frac{3c^2-\gamma}{c^2}\frac{\vec{\ell}\cdot\vec{m}}{h_1h_2}\big]n_1\\
=&-\phi\frac{pqh_1h_2}{2c}\big[\frac{c^2-\gamma}{2c^2}(|\vec{R}|^2+|\vec{S}|^2) - \frac{3c^2-\gamma}{c^2}\vec{R}\cdot\vec{S}\big]n_1\\
=&-\phi\frac{pqh_1h_2}{2c}\big[\frac{c^2-\gamma}{c^2}(|\mathbf{n}_t|^2+c^2|\mathbf{n}_x|^2) - \frac{3c^2-\gamma}{c^2}(|\mathbf{n}_t|^2-c^2|\mathbf{n}_x|^2)\big]n_1.\\
=&-2\phi\frac{pqh_1h_2}{2c}\big(-|\mathbf{n}_t|^2+(2c^2-\gamma)|\mathbf{n}_x|^2\big)n_1,
\end{aligned}
$$

which implies that the first equation in (1.1) holds in integral form. Similarly, $\mathbf{n}$ satisfies the second and third equations of system (1.1) in distributional sense.

5. Upper bound on energy

We convert the energy conservation (1.21) formally to the (X, Y) plane to look for conservation of energy.

Recall that the energy conservation law (1.21) can be written as (2.3) in terms of $|\vec{R}|$ and $|\vec{S}|$. By the variables (2.12), we rewrite the equation (2.3) as

$$
\left(\frac{1}{4h_1}+\frac{1}{4h_2}-\frac{1}{2}\right)_t - \left[\frac{c}{4}(\frac{1}{h_1}-\frac{1}{h_2})\right]_x = 0.
$$

It yields a closed form

$$
\left(\frac{1}{4h_1}+\frac{1}{4h_2}-\frac{1}{2}\right)dx + \left[\frac{c}{4}(\frac{1}{h_1}-\frac{1}{h_2})\right]dt
\tag{5.1}
$$

which, using the formula

$$
\begin{aligned}
dt &= t_X dX + t_Y dY = \tfrac{ph_1}{2c}dX + \tfrac{qh_2}{2c}dY\\
dx &= x_X dX + x_Y dY = \tfrac{ph_1}{2}dX - \tfrac{qh_2}{2}dY
\end{aligned}
\tag{5.2}
$$

can be written as

$$
\frac{p(1-h_1)}{4}dX - \frac{q(1-h_2)}{4}dY.
\tag{5.3}
$$

This is the energy form in the (X,Y) plane. The solutions $\mathbf{n}=\mathbf{n}(X,Y)$ constructed in Section 2 are *conservative*, in the sense that the integral of the form (5.3) along every Lipschitz continuous, closed curve in the X-Y plane is zero.

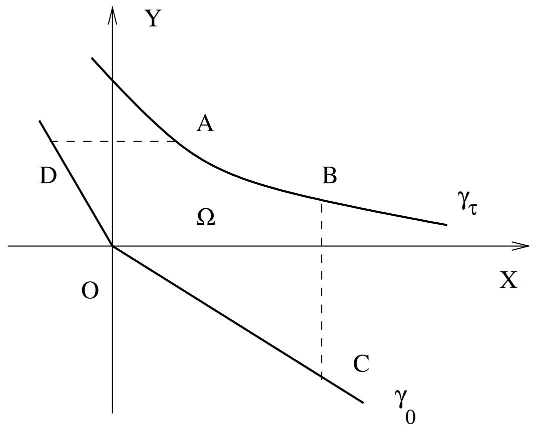

FIGURE 1. Energy conservation

We use (5.3) to establish that the energy of our solution is bounded, i.e., the energy inequality (1.26). Fix $\tau>0$, and $r>>1$. Define the set

$$\Omega := \Big\{(X,Y)\,;\ 0\le t(X,Y)\le\tau\,,\quad X\le r\,,\ Y\le r\Big\}. \tag{5.4}$$

See Figure 1, where segment AD is where $Y=r$ while segment BC is where $X=r$. By construction, the map $(X,Y)\mapsto(t,x)$ will act as follows:

$$A\mapsto(\tau,a)\,,\qquad B\mapsto(\tau,b)\,,\qquad C\mapsto(0,c)\,,\qquad D\mapsto(0,d)\,,$$

for some $a<b$ and $d<c$. Integrating the 1-form (5.3) along the boundary of Ω we obtain

$$\begin{aligned}
&\textstyle\int_{AB}\frac{p(1-h_1)}{4}\,dX-\frac{q(1-h_2)}{4}\,dY\\
&=\textstyle\int_{DC}\frac{p(1-h_1)}{4}\,dX-\frac{q(1-h_2)}{4}\,dY\ -\int_{DA}\frac{p(1-h_1)}{4}\,dX\\
&\quad\textstyle-\int_{CB}\frac{q(1-h_2)}{4}\,dY\\
&\le\textstyle\int_{DC}\frac{p(1-h_1)}{4}\,dX-\frac{q(1-h_2)}{4}\,dY\\
&=\textstyle\int_d^c\frac{1}{2}\Big[|\mathbf{n}_t|^2(0,x)+c^2\big(n_1(0,x)\big)\,|\mathbf{n}_x|^2(0,x)\Big]\,dx\,.
\end{aligned}\tag{5.5}$$

On the other hand, we use (5.2) to compute

$$\begin{aligned}
&\int_a^b\frac{1}{2}\Big[|\mathbf{n}_t|^2(\tau,x)+c^2\big(n_1(\tau,x)\big)\,|\mathbf{n}_x|^2(\tau,x)\Big]\,dx\\
&=\int_{AB\cap\{h_1\neq0\}}\frac{p(1-h_1)}{4}dX-\int_{AB\cap\{h_2\neq0\}}\frac{q(1-h_2)}{4}dY\le\mathcal{E}_0.
\end{aligned}\tag{5.6}$$

Notice that the last relation in (5.5) is satisfied as an equality, because at time $t=0$, along the curve γ_0 the variables h_1,h_2 never assume the value zero. The proof for the case $\tau<0$ is similar. Letting $r\to+\infty$ in (5.4), one has $a\to-\infty$, $b\to+\infty$. Therefore (5.5) and (5.6) together imply $\mathcal{E}(t)\le\mathcal{E}_0$, proving (1.26).

6. Regularity of trajectories

6.1. **Lipschitz continuity.** In this part, we first prove the Lipschitz continuity of the map $t \mapsto (n_1, n_2, n_3)(t, \cdot)$ in the L^2 distance, stated in (1.23). For any $h > 0$, we have

$$\mathbf{n}(t+h,x) - \mathbf{n}(t,x) = h \int_0^1 \mathbf{n}_t(t+\tau h, x)\, d\tau.$$

Thus

$$\|n_i(t+h,x) - n_i(t,x)\|_{L^2} \le h \int_0^1 \|n_{it}(t+\tau h,\cdot)\|_{L^2}\, d\tau \le h\sqrt{2\mathcal{E}_0}, \quad i = 1 \sim 3. \tag{6.1}$$

6.2. **Continuity of derivatives.** We prove the continuity of functions $t \mapsto (n_{1t}, n_{2t}, n_{3t})(t, \cdot)$ and $t \mapsto (n_{1x}, n_{1x}, n_{1x})(t, \cdot)$, as functions with values in L^p, $1 \le p < 2$. (To keep tradition, we use the exponent p here at the expense of repeating one of our primary variables.) This will complete the proof of Theorem 1.1.

We first consider the case where the initial data $(\mathbf{n}_0)_x$ and $\mathbf{n}_1$ are smooth with compact support. In this case, the solution $\mathbf{n} = \mathbf{n}(X, Y)$ remains smooth on the entire X-Y plane. Fix a time τ. We claim that

$$\frac{d}{dt}\mathbf{n}(t,\cdot)\Big|_{t=\tau} = \mathbf{n}_t(\tau,\cdot) \tag{6.2}$$

where

$$\mathbf{n}_t(\tau, x) := \mathbf{n}_X\, X_t + \mathbf{n}_Y\, Y_t = \frac{p\,\vec{\ell}}{2c}\,\frac{c}{ph_1} + \frac{q\vec{m}}{2c}\,\frac{c}{qh_2} = \frac{\vec{\ell}}{2h_1} + \frac{\vec{m}}{2h_2}\,. \tag{6.3}$$

Notice that (6.3) defines the values of $\mathbf{n}_t(\tau, \cdot)$ at almost every point $x \in \mathbb{R}$. By the inequality (1.26), we obtain

$$\int_{\mathbb{R}} \left|\mathbf{n}_t(\tau, x)\right|^2 dx \le 2\,\mathcal{E}(\tau) \le 2\,\mathcal{E}_0. \tag{6.4}$$

To prove (6.2), we consider the set

$$\Gamma_\tau := \{(X, Y)\,|\, t(X, Y) \le \tau\}, \tag{6.5}$$

and let γ_τ be its boundary. Let $\varepsilon > 0$ be given. There exist finitely many disjoint intervals $[a_i,\, b_i] \subset \mathbb{R}$, $i = 1, \ldots, N$, with the following property. Call A_i, B_i the points on γ_τ such that $x(A_i) = a_i$, $x(B_i) = b_i$. Then one has

$$\min\left\{h_1(P), h_2(P)\right\} < 2\varepsilon \tag{6.6}$$

at every point P on γ_τ contained in one of the arcs A_iB_i, while

$$h_1(P) > \varepsilon\,, \qquad h_2(P) > \varepsilon\,, \tag{6.7}$$

for every point P along γ_τ, not contained in any of the arcs A_iB_i. Call $J := \cup_{1\le i\le N}[a_i, b_i]$, $J' = \mathbb{R}\setminus J$, and notice that, as a function of the original variables, $\mathbf{n} = \mathbf{n}(t, x)$ is smooth in a neighborhood of the set $\{\tau\} \times J'$. Using Minkowski's inequality and the differentiability of $\mathbf{n}$ on J', we can write, for $i = 1 \sim 3$,

$$\begin{aligned} &\lim_{h\to 0} \ \tfrac{1}{h}\left(\int_{\mathbb{R}} \left|n_i(\tau+h,x) - n_i(\tau,x) - h\, n_{it}(\tau,x)\right|^p dx\right)^{1/p} \\ &\quad\le \lim_{h\to 0} \tfrac{1}{h}\left(\int_J \left|n_i(\tau+h,x) - n_i(\tau,x)\right|^p dx\right)^{1/p} \\ &\qquad + \left(\int_J \left|n_{it}(\tau,x)\right|^p dx\right)^{1/p}\,. \end{aligned} \tag{6.8}$$

We now provide an estimate on the measure of the "bad" set J:

$$\begin{aligned} \text{meas}\,(J) &= \int_J dx = \sum_i \int_{A_iB_i} \tfrac{ph_1}{2}\,dX - \tfrac{qh_2}{2}\,dY \\ &\le \tfrac{2\varepsilon}{1-2\varepsilon} \sum_i \int_{A_iB_i} \tfrac{p(1-h_1)}{2}\,dX - \tfrac{q(1-h_2)}{2}\,dY \\ &\le \tfrac{4\varepsilon}{1-2\varepsilon} \int_{\gamma_\tau} \tfrac{p(1-h_1)}{4}\,dX - \tfrac{q(1-h_2)}{4}\,dY \le \tfrac{4\varepsilon}{1-2\varepsilon}\,\mathcal{E}_0\,. \end{aligned} \tag{6.9}$$

Notice that $dt = 0$ on γ_τ, so the two parts of the integral are actually equal. Now choose $q = 2/(2-p)$ so that $\frac{p}{2} + \frac{1}{q} = 1$. Using Hölder's inequality with conjugate exponents $2/p$ and q, and recalling (6.1), we obtain for any $i = 1 \sim 3$

$$\begin{aligned} &\int_J \Big|n_i(\tau+h,x) - n_i(\tau,x)\Big|^p dx \\ &\le \text{meas}\,(J)^{1/q} \cdot \left(\int_J \Big|n_i(\tau+h,x) - n_i(\tau,x)\Big|^2 dx\right)^{p/2} \\ &\le \text{meas}\,(J)^{1/q} \cdot \left(\Big\|n_i(\tau+h,\cdot) - n_i(\tau,\cdot)\Big\|_{L^2}^2\right)^{p/2} \\ &\le \text{meas}\,(J)^{1/q} \cdot \Big(h^2\big[2\mathcal{E}_0\big]\Big)^{p/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\limsup_{h\to 0} \frac{1}{h} \left(\int_J \Big|n_i(\tau+h,x) - n_i(\tau,x)\Big|^p dx\right)^{1/p} \\ &\le \big[\frac{4\varepsilon}{1-2\varepsilon}\,\mathcal{E}_0\big]^{1/pq} \cdot \big[2\mathcal{E}_0\big]^{1/2}. \end{aligned} \tag{6.10}$$

In a similar way we estimate

$$\int_J \big|n_{it}(\tau,x)\big|^p dx \le \big[\text{meas}\,(J)\big]^{1/q} \cdot \left(\int_J \Big|n_{it}(\tau,x)\Big|^2 dx\right)^{p/2},$$

$$\left(\int_J \big|n_{it}(\tau,x)\big|^p dx\right)^{1/p} \le \text{meas}\,(J)^{1/pq} \cdot \big[2\mathcal{E}_0\big]^{p/2}. \tag{6.11}$$

Since $\varepsilon > 0$ is arbitrary, from (6.8), (6.10) and (6.11) we conclude

$$\lim_{h\to 0} \frac{1}{h} \left(\int_{\mathbb{R}} \Big|n_i(\tau+h,x) - n_i(\tau,x) - h\,n_{it}(\tau,x)\Big|^p dx\right)^{1/p} = 0. \tag{6.12}$$

The proof of continuity of the map $t \mapsto n_{it}$ is similar. Fix $\varepsilon > 0$. Consider the intervals $[a_i, b_i]$ as before. Since $\mathbf{n}$ is smooth on a neighborhood of $\{\tau\} \times J'$, it suffices to estimate

$$\begin{aligned} &\limsup_{h\to 0} \int \big|n_{it}(\tau+h,x) - n_{it}(\tau,x)\big|^p dx \\ &\le \limsup_{h\to 0} \int_J \big|n_{it}(\tau+h,x) - n_{it}(\tau,x)\big|^p dx \\ &\le \limsup_{h\to 0} \big[\text{meas}\,(J)\big]^{1/q} \cdot \left(\int_J \Big|n_{it}(\tau+h,\,x) - n_{it}(\tau,x)\Big|^2 dx\right)^{p/2} \\ &\le \limsup_{h\to 0} \big[\tfrac{4\varepsilon}{1-2\varepsilon}\mathcal{E}_0\big]^{1/q} \cdot \Big(\big\|n_{it}(\tau+h,\cdot)\big\|_{L^2} + \big\|n_{it}(\tau,\cdot)\big\|_{L^2}\Big)^p \\ &\le \big[\tfrac{4\varepsilon}{1-2\varepsilon}\mathcal{E}_0\big]^{1/q}\,\big[4\mathcal{E}_0\big]^p. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves continuity.

To extend the result to general initial data, such that $(\mathbf{n}_0)_x$, $\mathbf{n}_1 = \mathbf{n}_t|_{t=0} \in L^2$, we use Corollary 3.1 and consider a sequence of smooth initial data, with $(\mathbf{n}_0^\nu)_x, \mathbf{n}_1^\nu \in \mathbf{C}_c^\infty$, with $\mathbf{n}_0^\nu \to \mathbf{n}_0$ uniformly, $(\mathbf{n}_0^\nu)_x \to (\mathbf{n}_0)_x$ almost everywhere and in L^2, $\mathbf{n}_1^\nu \to \mathbf{n}_1$ almost everywhere and in L^2.

The continuity of the function $t \mapsto \mathbf{n}_x(t, \cdot)$ as maps with values in L^p, $1 \le p < 2$, is proved in an entirely similar way.

7. Energy conservation

This part is devoted to the proof of Theorem 1.3, stating that, in some sense, the total energy of the solution remains constant in time.

A key tool in our analysis is the *wave interaction potential*, defined in the smooth case by

$$\Lambda(t) := \frac{1}{16} \iint_{x>y} |\vec{R}|^2(t,x)\, |\vec{S}|^2(t,y)\, dxdy.$$

To define it in the general case, for any fixed time τ, we let $\mu_\tau = \mu_\tau^- + \mu_\tau^+$ be the positive measure on the real line defined as follows. In the smooth case,

$$\mu_\tau^-\big(\,(a,b)\,\big) = \frac{1}{4}\int_a^b |\vec{R}|^2(\tau,x)\,dx\,, \qquad \mu_\tau^+\big(\,(a,b)\,\big) = \frac{1}{4}\int_a^b |\vec{S}|^2(\tau,x)\,dx\,. \tag{7.1}$$

To define $\mu_\tau^\pm$ in the general case, let γ_τ be the boundary of the set

$$\Gamma_\tau := \{(X,Y)\,|\,t(X,Y) \le \tau\}, \tag{7.2}$$

and let γ_τ be its boundary. Given any open interval (a,b), let $A = (X_A, Y_A)$ and $B = (X_B, Y_B)$ be the points on γ_τ such that

$$x(A) = a\,, \quad X_P - Y_P \le X_A - Y_A \quad \text{for every point } P \in \gamma_\tau \text{ with } x(P) \le a\,,$$

$$x(B) = b\,, \quad X_P - Y_P \ge X_B - Y_B \quad \text{for every point } P \in \gamma_\tau \text{ with } x(P) \ge b\,.$$

Then

$$\mu_\tau\big(\,(a,b)\,\big) = \mu_\tau^-\big(\,(a,b)\,\big) + \mu_\tau^+\big(\,(a,b)\,\big)\,, \tag{7.3}$$

where

$$\mu_\tau^-\big(\,(a,b)\,\big) := \int_{AB} \frac{p(1-h_1)}{4}\,dX \quad \mu_\tau^+\big(\,(a,b)\,\big) := -\int_{AB} \frac{q(1-h_2)}{4}\,dY\,. \tag{7.4}$$

It is clear that μ_τ^-, μ_τ^+ are bounded, nonnegative measures, and $\mu_\tau(\mathbb{R}) = \mathcal{E}_0$, for all τ. The wave interaction potential is defined as

$$\Lambda(t) := (\mu_t^- \otimes \mu_t^+)\big\{(x,y)\,;\; x > y\big\}\,. \tag{7.5}$$

Lemma 7.1 (Bounded variation)**.** *The map $t \mapsto \Lambda(t)$ has locally bounded variation; i.e., there exists a one-sided Lipschitz constant L_0 such that*

$$\Lambda(t) - \Lambda(s) \ \le L_0 \cdot (t-s), \qquad\qquad t > s > 0\,.$$

Before we prove the lemma, let us first give a formal argument, valid when the solution $\mathbf{n}(t,x)$ remains smooth. Thanks to (2.8) and (2.10), we obtain

$$\begin{aligned} \partial_t|\vec{R}|^2 - \partial_x(c|\vec{R}|^2) &= A, \\ \partial_t|\vec{S}|^2 + \partial_x(c|\vec{S}|^2) &= -\,A, \end{aligned} \tag{7.6}$$

where

$$A = \frac{c'(n_1)}{2c(n_1)}\big(|\vec{R}|^2 S_1 - |\vec{S}|^2 R_1\big). \tag{7.7}$$

Then

$$\begin{aligned}\frac{d}{dt}[16\Lambda(t)] \leq & -\int 2c\,|\vec{R}|^2|\vec{S}|^2\,dx + \int \big(|\vec{S}|^2 + |\vec{R}|^2\big)\,dx \int A\,dx \\ \leq & -2C_L \int |\vec{R}|^2|\vec{S}|^2\,dx + 4\mathcal{E}_0 \left\|\frac{c'}{2c}\right\|_{L^\infty} \int \big||\vec{R}|^2 S_1 - R_1|\vec{S}|^2\big|\,dx\,.\end{aligned} \tag{7.8}$$

For each $\varepsilon > 0$ we have $|R_1| \leq \varepsilon^{-1/2} + \varepsilon^{1/2}\,|\vec{R}|^2$. Choosing $\varepsilon > 0$ such that

$$C_L \geq 4\mathcal{E}_0 \left\|\frac{c'}{2c}\right\|_{L^\infty} \cdot 2\sqrt{\varepsilon}\,,$$

we thus obtain

$$\frac{d}{dt}[16\Lambda(t)] \leq -C_L \int |\vec{R}|^2|\vec{S}|^2\,dx + \frac{16\,\mathcal{E}_0^2}{\sqrt{\varepsilon}} \left\|\frac{c'}{2c}\right\|_{L^\infty} \leq \frac{16\,\mathcal{E}_0^2}{\sqrt{\varepsilon}} \left\|\frac{c'}{2c}\right\|_{L^\infty}.$$

Hence, the map $t \mapsto \Lambda(t)$ has bounded variation on any bounded interval. It can be discontinuous, with downward jumps.

Proof. We reproduce the formal argument in terms of the variables X, Y. In particular, we reproduce (7.8) in integral form in the (X, Y) plane. By (2.21), for any $\varepsilon > 0$, there exists a constant κ_ε such that

$$m_1 \leq \varepsilon(1-h_2) + \kappa_\varepsilon h_2, \quad \ell_1 \leq \varepsilon(1-h_1) + \kappa_\varepsilon h_1.$$

Hence

$$|(1-h_1)m_1 - (1-h_2)\ell_1| \leq \varepsilon(1-h_1)(1-h_2) + \kappa_\varepsilon[(1-h_1)h_2 + (1-h_2)h_1]. \tag{7.9}$$

Fix $0 \leq s < t$. Consider the sets Γ_s, Γ_t as in (7.2) and define $\Gamma_{st} := \Gamma_t \backslash \Gamma_s$. Recall that

$$dx\,dt = \frac{pqh_1h_2}{2c}\,dX\,dY, \quad |\vec{R}|^2 = \frac{1-h_1}{h_1}, \quad |\vec{S}|^2 = \frac{1-h_2}{h_2}.$$

We can write

$$\begin{aligned}\textstyle\int_s^t \int_{-\infty}^{\infty} \frac{|\vec{R}|^2+|\vec{S}|^2}{4}\,dxdt \;&=\; (t-s)\mathcal{E}_0 \\ = \textstyle\iint_{\Gamma_{st}} \frac{1}{4}\left(\frac{1-h_1}{h_1} + \frac{1-h_2}{h_2}\right) \cdot \frac{pqh_1h_2}{2c}\,dXdY\,.&\end{aligned} \tag{7.10}$$

The first identity holds only for smooth solutions, but the second one is always valid. Recalling (5.1) and (7.4), and then using (7.9)-(7.10), we obtain

$$\begin{aligned}\Lambda(t) - \Lambda(s) \leq & -\iint_{\Gamma_{st}} \frac{1-h_1}{4}p \cdot \frac{1-h_2}{4}q\,dXdY \\ & + 4\mathcal{E}_0 \cdot \iint_{\Gamma_{st}} \frac{c'}{64c^2}pq|\,[(1-h_1)m_1 - (1-h_2)\ell_1]\,|\,dXdY \\ \leq & -\frac{1}{16}\iint_{\Gamma_{st}} (1-h_1)(1-h_2)\,pq\,dXdY \\ & + \mathcal{E}_0 \cdot \iint_{\Gamma_{st}} \frac{c'}{16c^2}pq\,\{\kappa_\varepsilon \cdot [(1-h_1)h_2 + (1-h_2)h_1] \\ & + \varepsilon(1-h_1)(1-h_2)\}\;dXdY \leq \kappa(t-s)\,,\end{aligned}$$

for a suitable constant κ. This proves the lemma. □

To prove Theorem 1.3, consider the three sets

$$\Omega_1 := \Big\{(X,Y)\,;\ h_1(X,Y)=0\,,\quad h_2(X,Y)\neq 0\,,\quad \big(c^2(n_1(X,Y))-\gamma\big)n_1(X,Y)\neq 0\Big\}\,,$$

$$\Omega_2 := \Big\{(X,Y)\,;\ h_2(X,Y)=0\,,\quad h_1(X,Y)\neq 0\,,\quad \big(c^2(n_1(X,Y))-\gamma\big)n_1(X,Y)\neq 0\Big\}\,,$$

$$\Omega_3 := \Big\{(X,Y)\,;\ h_1(X,Y)=0\,,\quad h_2(X,Y)= 0\,,\quad \big(c^2(n_1(X,Y))-\gamma\big)n_1(X,Y)\neq 0\Big\}\,.$$

Here $\big(c^2(n_1)-\gamma\big)n_1\neq 0$ is equivalent to $n_1\neq\pm 1$ or 0, since we assume that $\alpha\neq\gamma$.

From the equations (2.20), it follows that

$$\text{meas}\,(\Omega_1)=\text{meas}\,(\Omega_2)=0\,. \tag{7.11}$$

Indeed, $\partial_Y\ell_1\neq 0$ on Ω_1 since $h_1=0$ implies $\vec{\ell}=0$ by (2.21). Similarly $\partial_X m_1\neq 0$ on Ω_2 since $h_2=0$ implies $\vec{m}=0$. And, for any $i=1\sim 3$, $\ell_i\neq 0$ implies $h_1\neq 0$, $m_i\neq 0$ implies $h_2\neq 0$, by (2.12).

Let Ω_3^* be the set of Lebesgue points of Ω_3. We now show that

$$\text{meas}\Big(\big\{t(X,Y)\,;\ (X,Y)\in\Omega_3^*\big\}\Big)=0\,. \tag{7.12}$$

To prove (7.12), fix any $P^*=(X^*,Y^*)\in\Omega_3^*$ and let $\tau=t(P^*)$. We claim that

$$\limsup_{h,k\to 0+}\ \frac{\Lambda(\tau-h)-\Lambda(\tau+k)}{h+k}\ =\ +\infty \tag{7.13}$$

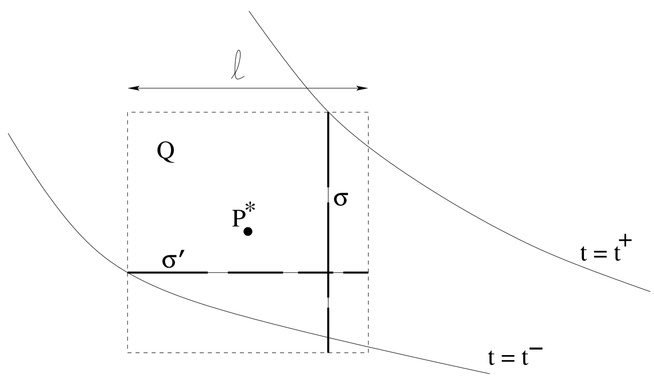

FIGURE 2. Lebesgue point

By assumption, for any $\varepsilon>0$ arbitrarily small we can find $\delta>0$ with the following property. For any square Q centered at P^* with side of length $\ell<\delta$, there exists a vertical segment σ and a horizontal segment σ', as in Figure 2, such that

$$\text{meas}\,\big(\Omega_3\cap\sigma\big)\geq(1-\varepsilon)\ell\,,\qquad\qquad \text{meas}\,\big(\Omega_3\cap\sigma'\big)\geq(1-\varepsilon)\ell\,, \tag{7.14}$$

Call

$$t^+ := \max\Big\{t(X,Y)\,;\ (X,Y)\in\sigma\cup\sigma'\Big\}\,,$$

$$t^- := \min\Big\{t(X,Y)\,;\ (X,Y)\in\sigma\cup\sigma'\Big\}\,.$$

Since $h_1 = h_2 = 0$ at nearly all points close to P^*, we can assume that the endpoints of the two segments σ, σ' are all in Ω_3. By integrating the equation for ℓ_1 from (2.20), we obtain

$$\begin{aligned}\int_\sigma \frac{q}{8c^3(n_1)}\big[(c^2(n_1)-\gamma)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\gamma)\vec{\ell}\cdot\vec{m}\big]n_1 \\ + \frac{c'(n_1)}{4c^2(n_1)}\ell_1 q(\ell_1 - m_1)dY = 0.\end{aligned} \tag{7.15}$$

Notice that $h_1, \vec{\ell}$ are Lipschitz in Y and $h_1 = 0$ implies $\vec{\ell} = 0$, and they are zero on σ on a set with measure greater than $(1-\varepsilon)\ell$. So we obtain

$$\int_\sigma (h_1 f_1 + \ell_1 f_2 + \ell_2 f_3 + \ell_3 f_4)\, dY = O(1)(\varepsilon\ell)^2 \tag{7.16}$$

for any bounded functions $f_1 \sim f_4$. Thus we obtain from (7.15)(7.16) that

$$\int_\sigma \frac{\big(c^2(n_1)-\gamma\big)n_1}{8c^3(n_1)} q h_2\, dY = O(1)(\varepsilon\ell)^2. \tag{7.17}$$

By (5.2), we have

$$\int_\sigma t_Y\, dY = \int_\sigma \frac{qh_2}{2c}\, dY.$$

Assume without loss of generality that $\big(c^2(n_1)-\gamma\big)n_1 > C_3 > 0$, at the point P^*. By (7.17), we obtain

$$\int_\sigma t_Y\, dY = O(1)(\varepsilon\ell)^2.$$

Similarly we can estimate the growth of t in the X-direction. Combining them, we obtain

$$t^+ - t^- \le O(1)(\varepsilon\ell)^2. \tag{7.18}$$

On the other hand,

$$\Lambda(t^-) - \Lambda(t^+) \ge C_1(1-\varepsilon)^2\ell^2 - C_2(t^+ - t^-) \tag{7.19}$$

for some constant $C_1 > 0, C_2 > 0$. Since $\varepsilon > 0$ is arbitrary, this implies (7.13).

Recalling that the map $t \mapsto \Lambda$ has bounded variation, from (7.13) it follows (7.12).

We now observe that the singular part of μ_τ is nontrivial only if the set

$$\big\{P \in \gamma_\tau\,;\ h_1(P) = 0 \text{ or } h_2(P) = 0\big\}$$

has positive 1-dimensional measure. By the previous analysis, restricted to the region where $\big(c^2(n_1)-\gamma\big)n_1 \ne 0$, i.e. $n_1 \ne \pm 1$ or 0, this can happen only for a set of times having zero measure.

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