Shock formation in the compressible Euler equations and related systems

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Abstract

We prove shock formation results for the compressible Euler equations and related systems of conservation laws in one space dimension, or three dimensions with spherical symmetry. We establish an L^∞ bound for C^1 solutions of the one-D Euler equations, and use this to improve recent shock formation results of the authors. We prove analogous shock formation results for one-D MHD with orthogonal magnetic field, and for compressible flow in a variable area duct, which has as a special case spherically symmetric three dimensional flow on the exterior of a ball.

1   Introduction

One of the defining features of systems of hyperbolic conservation laws,

$$u_t + f(u)_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \ u \in \mathbb{R}^n, \tag{1.1}$$

in the absence of regularizing dissipative effects such as viscosity and heat loss, is the blowup of gradients and subsequent formation of shock waves.

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Once a shock forms, discontinuous weak solutions must be considered, and the corresponding loss of regularity makes the systems hard to analyze.

We are concerned with the conditions under which a shock forms in a classical C^1 solution of (1.1). For pairs of equations ($n = 2$), Lax showed in [13, 14] that shocks always form for data which is anywhere compressive. In work initiated by John, this was extended to larger systems, given restrictions on the size of the initial data [11, 16, 21]. In particular, all waves in those solutions are assumed to be weak. One of our goals is to remove these severe restrictions on the initial data, which includes allowing for large waves in the solution. We restrict our attention to systems arising from physical problems because of the variety of unstable phenomena which can affect general 3×3 systems when the data is allowed to be large [9, 10, 27, 28, 30].

Recently, the authors have obtained large data results for the 3×3 compressible Euler equations [2, 4]. In all of these works, (1.1) is differentiated and manipulated to get a system of PDEs with quadratic nonlinearities for the evolution of gradients along characteristics. For two equations, these PDEs decouple to yield a pair of Riccati type ODEs along characteristics; the quadratic nonlinearity then implies that the derivatives must blow up in finite time [13, 14]. For general systems, if the data has small total variation, the nonlinear waves separate in finite time, after which they are essentially decoupled and genuine nonlinearity in each field causes gradients to blow up by the same mechanism.

In [2, 4], we consider the compressible Euler equations, and make a series of nonlinear changes of variables to get an uncoupled pair of quadratic ODEs for the genuinely nonlinear fields, which exhibit the same blowup mechanism. Although there are no restrictions on the size of the initial data, the results of [2, 4] rely on assumed global bounds for the state variables. In this paper, we first remove this restriction by obtaining an *a priori* estimate on the state variables for C^1 solutions of the Euler equations, yielding a condition on only the initial data which guarantees blowup of gradients. See [19] for a description of attempts to obtain a stronger L^∞ bound and global existence which were unsuccessful.

We then extend the method to larger systems of physical interest. We show similar gradient blowup results for one-dimensional magnetohydrodynamics (MHD) with orthogonal magnetic field, and compressible flow in a variable area duct. As a special case, we obtain a gradient blowup result for spherically symmetric three-dimensional Euler flow on the exterior of a ball. These blowup results again rely on global bounds for the state variables, but in view of our obtained bounds for the compressible Euler equations, we expect that this is just a technical assumption. Indeed, we

note that all known cases of blowup of state variables occur in non-physical systems [9, 10, 27, 28, 30].

The mechanism for gradient blowup and shock formation is the steepening of compressions due to genuine nonlinearity: if the wavespeed is larger behind the wave, it will eventually break. For 2×2 systems (or diagonal systems such as chromatography), the different wave families weakly decouple and all compressions will collapse into shocks. However, in 3×3 systems, and especially for large data, the wave interactions between different families influence this process and can delay the onset of shocks [18, 20, 25]. In the context of Euler, the contacts act as partial reflectors of waves and with care one can set up solutions in which compressions are cancelled before they focus [25, 5].

Our conditions which guarantee blowup are expressed in terms of global bounds for the variables, so we must ensure that the state does not leave some compact set. In particular, for a γ-law gas, the vacuum must be avoided. Moreover, when $1 < \gamma < 3$, our estimates depend on a lower bound for the density as well as on the upper bound. This is also true in the isentropic case treated by Lax.

The paper is organized as follows: in Section 2, we obtain *a priori* L^∞ bounds for the density and velocity in the one-D compressible Euler equations. We then use this to stregthen the gradient blowup result of [2]. In Section 3, we consider one-D MHD with orthogonal magnetic field, derive Riccati type equations for gradients, and use these to obtain a stronger singularity formation result than that of [4]. In Section 4, we similarly derive Riccati type equations and present a shock formation theorem for compressible Euler flow in a variable area duct. This includes the special case of three-D spherically symmetric flow in the exterior of a ball.

2   Compressible Euler equations

We begin by considering the initial value problem for the compressible Euler equations in a Lagrangian frame in one space dimension,

$$\tau_t - u_x = 0, \tag{2.1}$$

$$u_t + p_x = 0, \tag{2.2}$$

$$(\frac{1}{2}u^2 + e)_t + (u\,p)_x = 0, \tag{2.3}$$

where ρ is the density, $\tau = \rho^{-1}$ is the specific volume, p is the pressure, u is the velocity, e is the internal energy, and x is the material coordinate. The

system is closed by the Second Law of Thermodynamics,

$$T\,dS = de + p\,d\tau, \tag{2.4}$$

where S is the entropy and T the temperature. For C^1 solutions, it follows that (2.3) is equivalent to the "entropy equation"

$$S_t = 0. \tag{2.5}$$

When the entropy is constant, the flow is isentropic, and (2.1) and (2.2) become a closed system, known as the p-system [24].

We assume the gas is ideal polytropic, so that

$$p\,\tau = R\,T \quad \text{and} \quad e = c_\tau\,T$$

with ideal gas constant R, and specific heat c_τ; this implies

$$p = K\,e^{\frac{S}{c_\tau}}\,\tau^{-\gamma} \tag{2.6}$$

with adiabatic gas constant $\gamma = \frac{R}{c_\tau} + 1$, see [6]. The nonlinear Lagrangian sound speed is

$$c := \sqrt{-p_\tau} = \sqrt{K\,\gamma}\,\tau^{-\frac{\gamma+1}{2}}\,e^{\frac{S}{2c_\tau}}. \tag{2.7}$$

2.1 Coordinates

We use the coordinates introduced by Temple and Young in [25]. Define new variables m and η for S and τ, by

$$m := e^{\frac{S}{2c_\tau}} > 0, \tag{2.8}$$

and, referring to (2.7),

$$\eta := \int_\tau^\infty \frac{c}{m}\,d\tau = \frac{2\sqrt{K\gamma}}{\gamma-1}\,\tau^{-\frac{\gamma-1}{2}} > 0. \tag{2.9}$$

It follows that

$$\begin{aligned}
\tau &= K_\tau\,\eta^{-\frac{2}{\gamma-1}}, \\
p &= K_p\,m^2\,\eta^{\frac{2\gamma}{\gamma-1}}, \\
c &= c(\eta, m) = K_c\,m\,\eta^{\frac{\gamma+1}{\gamma-1}},
\end{aligned} \tag{2.10}$$

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where K_τ, K_p and K_c are positive constants given by

$$K_\tau = \left(\tfrac{2\sqrt{K\gamma}}{\gamma-1}\right)^{\frac{2}{\gamma-1}}, \quad K_p = K\,K_\tau^{-\gamma}, \quad \text{and} \quad K_c = \sqrt{K\gamma}\,K_\tau^{-\frac{\gamma+1}{2}}, \tag{2.11}$$

so that also

$$K_p = \tfrac{\gamma-1}{2\gamma}K_c \quad \text{and} \quad K_\tau K_c = \tfrac{\gamma-1}{2}. \tag{2.12}$$

In these coordinates, for C^1 solutions, equations (2.1)–(2.3) are equivalent to

$$\eta_t + \frac{c}{m}\,u_x = 0, \tag{2.13}$$

$$u_t + m\,c\,\eta_x + 2\frac{p}{m}\,m_x = 0, \tag{2.14}$$

$$m_t = 0, \tag{2.15}$$

the last equation being (2.5), which is equivalent to (2.3). Note that, while the solution remains C^1, $m = m(x)$ is given by the initial data and can be regarded as stationary.

2.2 L^∞ bounds for C^1 solutions

Our first goal is to provide a uniform upper bound for the density ρ and velocity $|u|$, depending only on the initial data, for C^1 solutions of (2.1)–(2.3). Denote the Riemann invariants by

$$r := u - m\,\eta, \qquad s := u + m\,\eta. \tag{2.16}$$

For isentropic C^1 solutions (m constant) the Riemann invariants s and r are constants along forward and backward characteristics, respectively. However, for general non-isentropic flow, s and r vary along characteristics.

The forward and backward characteristics are described by

$$\frac{dx}{dt} = c \quad \text{and} \quad \frac{dx}{dt} = -c,$$

and we denote the corresponding directional derivatives along these by

$$\partial_+ := \tfrac{\partial}{\partial t} + c\,\tfrac{\partial}{\partial x} \quad \text{and} \quad \partial_- := \tfrac{\partial}{\partial t} - c\,\tfrac{\partial}{\partial x},$$

respectively. Using (2.16), (2.10) and (2.12), equations (2.13) and (2.14) become

$$\partial_+ s = \frac{1}{2\gamma}\frac{\partial_+ m}{m}\,(s - r), \tag{2.17}$$

$$\partial_- r = \frac{1}{2\gamma}\frac{\partial_- m}{m}\,(r - s), \tag{2.18}$$

while by (2.3), we have

$$\partial_+ m = c\, m'(x), \quad \text{and} \quad \partial_- m = -c\, m'(x). \tag{2.19}$$

Applying an integrating factor and setting

$$\widetilde{s} = m^{-\frac{1}{2\gamma}}\, s \quad \text{and} \quad \widetilde{r} = m^{-\frac{1}{2\gamma}}\, r,$$

we obtain the system

$$\partial_+ \widetilde{s} = -\frac{1}{2\gamma}\frac{\partial_+ m}{m}\, \widetilde{r}, \tag{2.20}$$

$$\partial_- \widetilde{r} = -\frac{1}{2\gamma}\frac{\partial_- m}{m}\, \widetilde{s}. \tag{2.21}$$

We obtain upper bounds for C^1 solutions of (2.1)–(2.3) by bounding $|\widetilde{s}|$ and $|\widetilde{r}|$. We assume that the initial entropy $S(x)$ is C^1 and has finite total variation, so that

$$V := \frac{1}{2c_\tau}\int_{-\infty}^{+\infty} |S'(x)|\, dx = \int_{-\infty}^{+\infty} \frac{|m'(x)|}{m(x)}\, dx < \infty, \tag{2.22}$$

while also, by (2.8),

$$0 < M_L < m(\cdot) < M_U, \tag{2.23}$$

for some constants M_L and M_U. Also, since ρ and $|u|$ are bounded initially, there exist positive constants M_s and M_r, such that, in the initial data,

$$|\widetilde{s}_0(\cdot)| < M_s \quad \text{and} \quad |\widetilde{r}_0(\cdot)| < M_r. \tag{2.24}$$

We define two useful constants by

$$N_1 := M_s + \overline{V}\, M_r + \overline{V}\, (\overline{V}\, M_s + \overline{V}^2\, M_r)\, e^{\overline{V}^2},$$

$$N_2 := M_r + \overline{V}\, M_s + \overline{V}\, (\overline{V}\, M_r + \overline{V}^2\, M_s)\, e^{\overline{V}^2},$$

where $\overline{V} := \frac{V}{2\gamma}$, which clearly depend only on the initial data.

Theorem 2.1. *For C^1 solutions of (2.1)–(2.3), we have the* a priori *bounds*

$$|u| \le \frac{N_1 + N_2}{2} M_U^{\frac{1}{2\gamma}} \quad and \quad \rho \le \frac{N_1 + N_2}{2} M_L^{\frac{1}{2\gamma}-1},$$

Note that the density ρ need not be bounded away from zero, as vacuums can form in infinite time [26, 5]. The theorem is an immediate consequence of the following estimate.

Lemma 2.2. *For a given point (x_1, t_1), suppose the solution is C^1 in the characteristic triangle bounded by the forward and backward characteristics through (x_1, t_1) and the line $t = 0$. Then*

$$|\widetilde{s}(x_1, t_1)| \leq N_1 \quad and \quad |\widetilde{r}(x_1, t_1)| \leq N_2.$$

Proof. We prove the bound on $\widetilde{s}(x_1, t_1)$; the bound of $\widetilde{r}(x_1, t_1)$ is obtained similarly. Referring to Figure 1, we denote the forward and backward characteristics through a point (x_*, t_*) by

$$\overrightarrow{\mathcal{L}}_{x_*} = \{(x, \overrightarrow{t}(x)) \mid x \leq x_*\} = \{(\overrightarrow{x}(t), t) \mid t \leq t_*\} \quad \text{and}$$

$$\overleftarrow{\mathcal{L}}_{x_*} = \{(x, \overleftarrow{t}(x)) \mid x \geq x_*\} = \{(\overleftarrow{x}(t), t) \mid t \leq t_*\},$$

respectively, parameterized by x or t, as convenient.

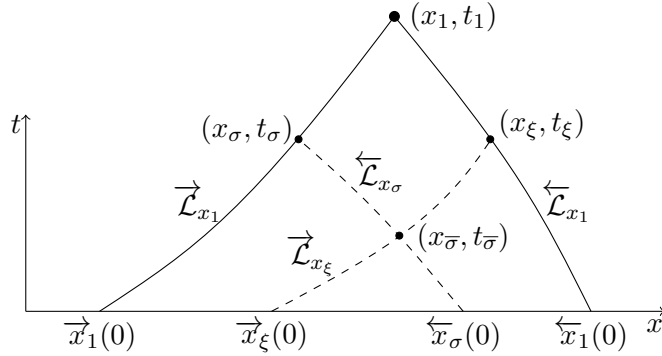Figure 1: Characteristic triangle

We integrate (2.20) along the forward characteristic $\overrightarrow{\mathcal{L}}_{x_1}$ from $(\overrightarrow{x}_1(0), 0)$ to (x_1, t_1), with a change of integration variable, and use x_σ as the new parameter. We then integrate (2.21) along the backward characteristics $\overleftarrow{\mathcal{L}}_{x_\sigma}$, from 0 to t_σ, for each $(x_\sigma, t_\sigma) \in \overrightarrow{\mathcal{L}}_{x_1}$, and again change variables. We have

$$\widetilde{s}(x_1, t_1) = \widetilde{s}_0(\overrightarrow{x}_1(0)) - \frac{1}{2\gamma} \int_{\overrightarrow{x}_1(0)}^{x_1} \frac{m'}{m}(x_\sigma) \widetilde{r}(x_\sigma, \overrightarrow{t}(x_\sigma)) \, dx_\sigma,$$

where we have used (2.19), and similarly

$$\widetilde{r}(x_\sigma, t_\sigma) = \widetilde{r}_0(\overleftarrow{x}_\sigma(0)) - \frac{1}{2\gamma} \int_{\overleftarrow{x}_\sigma(0)}^{x_\sigma} \frac{m'}{m}(x) \widetilde{s}(x, \overleftarrow{t}_\sigma(x)) \, dx.$$

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Here $\nearrow$ and $\nwarrow$ indicate the direction of the characteristic along which the integration is performed. Combining these, we get

$$\widetilde{s}(x_1, t_1) = \widetilde{s}_0(\overrightarrow{x}_1(0)) - \frac{1}{2\gamma} \int_{\overrightarrow{x}_1(0)}^{x_1} \frac{m'}{m}(x_\sigma)\, \widetilde{r}_0(\overleftarrow{x}_\sigma(0))\, dx_\sigma \qquad (2.25)$$

$$+ \frac{1}{4\gamma^2} \int_{\overrightarrow{x}_1(0)}^{x_1} \frac{m'}{m}(x_\sigma) \left(\int_{\overleftarrow{x}_\sigma(0)}^{x_\sigma} \frac{m'}{m}(x)\, \widetilde{s}(x, \overleftarrow{t}_\sigma(x))\, dx \right) dx_\sigma.$$

The first two terms can be estimated by our initial bounds, and we will apply a Gronwall inequality to estimate the third term.

Equation (2.25) continues to hold for any point (x_ξ, t_ξ) on the backward characteristic $\overleftarrow{\mathcal{L}}_{x_1}$, with x_1 replaced by x_ξ. We multiply (2.25) by $\frac{\partial_+ m}{m}(x_\xi)$, take absolute values, integrate in time along the backward characteristic $\overleftarrow{\mathcal{L}}_{x_1}$, and change to spatial variables. Doing this, we get

$$\int_{x_1}^{\overleftarrow{x}_1(0)} \left| \widetilde{s}(x_\xi, t_\xi)) \right| d\mu(x_\xi) \leq \int_{x_1}^{\overleftarrow{x}_1(0)} \left| \widetilde{s}_0(\overrightarrow{x}_\xi(0)) \right| d\mu(x_\xi)$$

$$+ \frac{1}{2\gamma} \int_{x_1}^{\overleftarrow{x}_1(0)} \int_{\overrightarrow{x}_\xi(0)}^{x_\xi} \left| \widetilde{r}_0(\overleftarrow{x}_{\overline{\sigma}}(0)) \right| d\mu(x_{\overline{\sigma}})\, d\mu(x_\xi) \qquad (2.26)$$

$$+ \frac{1}{4\gamma^2} \int_{x_1}^{\overleftarrow{x}_1(0)} \int_{\overrightarrow{x}_\xi(0)}^{x_\xi} \int_{x_{\overline{\sigma}}}^{\overleftarrow{x}_{\overline{\sigma}}(0)} \left| \widetilde{s}(x, \overleftarrow{t}_{\overline{\sigma}}(x)) \right| d\mu(x)\, d\mu(x_{\overline{\sigma}})\, d\mu(x_\xi),$$

where $d\mu$ is the Stieltjes measure

$$d\mu(x) := \left| \frac{m'}{m}(x) \right| dx,$$

which has total mass at most V along any characteristic. Again the first two terms of (2.26) can be estimated by the initial data, so we focus on the last term.

For (x_σ, t_σ) on the forward characteristic $\overrightarrow{\mathcal{L}}_{x_1}$, set

$$F(x_\sigma) := \int_{x_\sigma}^{\overleftarrow{x}_\sigma(0)} \left| \widetilde{s}(x, \overleftarrow{t}_\sigma(x)) \right| d\mu(x), \qquad (2.27)$$

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where the integral is along the backward characteristic $\overleftarrow{\mathcal{L}}_{x_\sigma}$. If the backward characteristic through $(x, \overleftarrow{t}_{\overline{\sigma}}(x))$ is extended up to $\overrightarrow{\mathcal{L}}_{x_1}$, the innermost integral in (2.26) is estimated by

$$\int_{x_{\overline{\sigma}}}^{\overleftarrow{x}_{\overline{\sigma}}(0)} |\widetilde{s}(x, \overleftarrow{t}_{\overline{\sigma}}(x))| \, d\mu(x) \le F(x_\sigma),$$

provided $(x_{\overline{\sigma}}, t_{\overline{\sigma}}) \in \overleftarrow{\mathcal{L}}_{x_\sigma}$, so that also $\overleftarrow{x}_{\overline{\sigma}}(0) = \overleftarrow{x}_\sigma(0)$, and we change the order of the other two integrations. Using (2.27) with (2.22) and (2.24), it follows that (2.26) yields

$$F(x_1) \le V \, M_s + \frac{1}{2\gamma} V^2 \, M_r + \frac{1}{4\gamma^2} \int_{\overrightarrow{x}_1(0)}^{x_1} \int_{x_\sigma}^{\overleftarrow{x}_\sigma(0)} F(x_\sigma) \, d\mu(\overline{x}) \, d\mu(x_\sigma)$$

$$\le V \, M_s + \frac{1}{2\gamma} V^2 \, M_r + \frac{1}{4\gamma^2} V \int_{\overrightarrow{x}_1(0)}^{x_1} F(x_\sigma) \, d\mu(x_\sigma).$$

We now use Gronwall's inequality to get

$$F(x_1) \le \left(V \, M_s + \tfrac{1}{2\gamma} V^2 \, M_r \right) e^{\frac{V^2}{4\gamma^2}} =: \hat{N}_1.$$

It follows similarly that for $(x_\sigma, t_\sigma) \in \overrightarrow{\mathcal{L}}_{x_1}$, we have $F(x_\sigma) \le \hat{N}_1$. Finally, using this estimate in (2.25), we get

$$|\widetilde{s}(x_1, t_1)| \le M_s + \tfrac{1}{2\gamma} V \, M_r + \tfrac{1}{4\gamma^2} V \, \hat{N}_1 = N_1,$$

and the proof is complete. $\qquad\qquad\square$

2.3   Singularity formation

In [2], we introduce gradient variables

$$y = m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}} \eta^{\frac{\gamma+1}{2(\gamma-1)}} \left((u + m\,\eta)_x - \tfrac{2}{3\gamma-1} m_x\,\eta \right) \quad \text{and}$$

$$q = m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}} \eta^{\frac{\gamma+1}{2(\gamma-1)}} \left((u - m\,\eta)_x + \tfrac{2}{3\gamma-1} m_x\,\eta \right), \tag{2.28}$$

and derive Riccati type equations for their evolution.

Lemma 2.3. *[2] For C^2 solutions of (2.1)–(2.3), we have*

$$\partial_+ y = a_0 + a_2\, y^2,$$
$$\partial_- q = a_0 + a_2\, q^2, \tag{2.29}$$

where

$$a_0 = \frac{K_c}{\gamma}\left[\frac{\gamma-1}{3\gamma-1}\, m\, m_{xx} - \frac{(3\gamma+1)(\gamma-1)}{(3\gamma-1)^2}\, m_x^2\right] m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}\, \eta^{\frac{3(\gamma+1)}{2(\gamma-1)}+1},$$
$$a_2 = -K_c\, \frac{\gamma+1}{2(\gamma-1)}\, m^{\frac{3(3-\gamma)}{2(3\gamma-1)}}\, \eta^{\frac{\gamma+1}{2(\gamma-1)}-1} < 0. \tag{2.30}$$

Furthermore,

$$|y|\ or\ |q| \to \infty \quad iff \quad |u_x|\ or\ |\tau_x| \to \infty. \tag{2.31}$$

Similar equations are derived independently in [18]. Using these equations, a singularity formation result was proved in [2]. By applying Lemma 2.2, we can improve that result to obtain our main theorem on the Euler equations.

Theorem 2.4. *Assume the initial data are C^2 with entropy having bounded variation, and suppose there is a positive constant M_* such that the initial entropy satisfies $|m''(x)| < M_*$. If $1 < \gamma < 3$, we also assume that the density has a positive global lower bound. There exists positive constant N depending only on the initial data, such that, if the initial data satisfies*

$$\inf \{\, y(\cdot, 0),\ q(\cdot, 0)\, \} < -N, \tag{2.32}$$

then $|u_x|$ and/or $|\tau_x|$ blow up in finite time.

This result is an extension of Lax's singularity formation result in [13] for 2×2 strictly hyperbolic systems. We recover the isentropic case by taking m constant, so $a_0 = 0$. The conditions which guarantee blowup are expressed in terms of global bounds for the variables, so we must ensure that the state does not leave some compact set, and, in particular, the vacuum must be avoided. When $1 < \gamma < 3$, our estimates depend on a lower bound for the density as well as on the upper bound. This is also true in the isentropic case treated by Lax. However, when $1 < \gamma < 3$, we expect a more refined analysis to show that even if the solution approaches vacuum, the gradient still blows up in finite time [29].

Proof. By Theorem 2.1 and (2.9), we know η has a global upper bound depending only on the initial data, denoted E_U.

By (2.30), it is easy to calculate that, if $a_0 \geq 0$,

$$\sqrt{-\frac{a_0}{a_2}} = \sqrt{\frac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}\left(m\,m_{xx} - \frac{3\gamma+1}{3\gamma-1}\,m_x^2\right)}\,\eta^{\frac{\gamma+1}{2(\gamma-1)}+1}\,m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}},$$

which implies the uniform bound $\sqrt{-a_0/a_2} \leq N$, where

$$N := \begin{cases} \sqrt{\frac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}}\,M_*\,E_U^{\frac{3\gamma-1}{2(\gamma-1)}}\,M_L^{\frac{3\gamma-5}{3\gamma-1}}, & 1 < \gamma \leq 5/3, \\ \sqrt{\frac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}}\,M_*\,E_U^{\frac{3\gamma-1}{2(\gamma-1)}}\,M_U^{\frac{3\gamma-5}{3\gamma-1}}, & \gamma \geq 5/3. \end{cases} \tag{2.33}$$

It follows that if $y < -N$, then $a_0 + a_2\,y^2 < 0$.

Now suppose that (2.32) holds. Then there exist $\varepsilon > 0$ and x_0 such that

$$y(x_0, 0) < -(1+\varepsilon)\,N,$$

say. Now consider the forward characteristic starting at $(x_0, 0)$. By (2.29), along this characteristic (parametrized by t) we have $\partial_+ y < 0$, so also

$$y(t) < -(1+\varepsilon)\,N \quad \text{for} \quad t \geq 0,$$

which in turn implies

$$a_0 + a_2\,\frac{y(t)^2}{(1+\varepsilon)^2} < 0$$

for all $t \geq 0$. Now (2.29) implies

$$\partial_+ y = a_0 + a_2\,y^2 < (1 - \tfrac{1}{(1+\varepsilon)^2})\,a_2\,y^2 < 0,$$

since $a_2 < 0$. Integrating, we get

$$\frac{1}{y(t)} \geq \frac{1}{y(0)} - \int_0^t (1 - \tfrac{1}{(1+\varepsilon)^2})\,a_2\,dt, \tag{2.34}$$

where the integral is along the forward characteristic. By (2.30), when $\gamma \geq 3$, a_2 is negative and bounded above, so the right hand side of (2.34) approaches zero in finite time. This implies that $y(t)$ approaches $-\infty$ in finite time, so that $|\tau_x|$ and/or $|u_x|$ blow up.

When $1 < \gamma < 3$, and assuming η has a positive (global) lower bound along the characteristic, then a_2 is again negative and bounded above, so the result follows similarly. It is clear that the same argument holds along a backward characteristic if $\inf q < -N$. $\qquad \square$

3 Magnetohydrodynamics

3.1 Prior results for generalized Euler equations

We recall the results of [4] for the generalized p-system (which includes the smooth Euler equations) in one space dimension,

$$\tau_t - u_x = 0,$$
$$u_t + P(\tau, x)_x = 0, \tag{3.1}$$

where $P(\tau, x)$ is a C^3 function of $\tau > 0$ and x, satisfying

$$P_\tau < 0 \quad \text{and} \quad P_{\tau\tau} > 0,$$

so that the system is hyperbolic with wavespeed

$$c := c(\tau, x) = \sqrt{-P_\tau}.$$

It is convenient to introduce new variables (h, μ) for (τ, x), by setting

$$h(\tau, x) := \int_\tau^{\tau^*} c \, d\tau = \int_\tau^{\tau^*} \sqrt{-P_\tau} \, d\tau \quad \text{and} \quad \mu = x,$$

where τ^* is a constant or infinity. It follows that

$$h_\tau = -c, \quad \tau_h = -\frac{1}{c}, \quad \text{and} \quad P_h = c, \tag{3.2}$$

and furthermore, for any C^1 function $f(\tau, x)$,

$$f_h = -\frac{f_\tau}{c} \quad \text{and} \quad f_\mu = \frac{h_x}{c} f_\tau + f_x. \tag{3.3}$$

We define gradient variables by

$$y := \sqrt{c}\,(u + h)_x + \frac{P_\mu}{\sqrt{c}} - I,$$
$$q := \sqrt{c}\,(u - h)_x - \frac{P_\mu}{\sqrt{c}} + I, \tag{3.4}$$

where

$$I = I(h, \mu) = \int_{h_0}^h \frac{1}{2} \sqrt{c} \left(\frac{P_\mu}{c}\right)_h dh, \tag{3.5}$$

and h_0 is a constant.

Lemma 3.1. *[4] For C^2 solutions of (3.1), we have*

$$\partial_+ y = a_0 + a_1\, y + a_2\, y^2,$$
$$\partial_- q = a_0 - a_1\, q + a_2\, q^2, \tag{3.6}$$

where

$$a_0 = -c\, I_\mu + \frac{1}{2}\, \sqrt{c}\, \left(\frac{P_\mu}{c}\right)_h\, P_\mu - c\, \left(\frac{P_\mu}{c}\right)_h\, I - \frac{c_h}{2\sqrt{c}}\, I^2,$$
$$a_1 = -\left(2\, \sqrt{c}\, I\right)_h = -c\, \left(\frac{P_\mu}{c}\right)_h - \frac{c_h}{\sqrt{c}}\, I, \tag{3.7}$$
$$a_2 = -\frac{c_h}{2\sqrt{c}} < 0.$$

For arbitrarily given positive constants A_i and B_i, we denote by $\mathcal{K}$ the compact set whose interior $\mathcal{K}^o$ is given by

$$|h| < B_1, \quad A_2 < c < B_2, \quad A_3 < c_h < B_3, \quad |c_\mu| < B_4$$
$$|c_{\mu\mu}| < B_5, \quad |c_{h\mu}| < B_6, \quad |p_\mu| < B_7, \quad |p_{\mu\mu}| < B_8.$$

Lemma 3.2. *[4] There exists a constant $\widetilde{N} > 0$ depending only on $\mathcal{K}$, such that, if the C^2 initial data of (3.1) satisfy*

$$(u(x,0),\, \tau(x,0)) \in \mathcal{K}^o, \quad \text{for all} \quad x,$$

and if

$$\inf\{\, y(\cdot,0),\ q(\cdot,0)\,\} < -\widetilde{N},$$

then there exists $T_* = T_*(\mathcal{K}, \widetilde{N})$ such that either

$$\max\{|u_x|, |\tau_x|\} \to \infty \quad \text{as} \quad t \to T_*,$$

or *there is some point (x_b, t_b) with $t_b \leq T_*$ such that*

$$(u(x_b, t_b),\, \tau(x_b, t_b)) \in \partial\mathcal{K}.$$

3.2 Refinements for MHD

In this section, we apply and upgrade Lemma 3.2 for one-D Magnetohydrodynamics. The system models the motion of a smooth compressible fluid

coupled to a magnetic field $H = (H_1, H_2, H_3)$. In a Lagrangian frame in one space dimension, C^1 solutions satisfy the quasilinear system

$$\frac{\partial \tau}{\partial t} - \frac{\partial u_1}{\partial x} = 0,$$

$$\frac{\partial H_j}{\partial t} + \rho \, H_j \, \frac{\partial u_1}{\partial x} - \rho \, H_1 \, \frac{\partial u_j}{\partial x} = 0, \quad j = 2, 3$$

$$\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x} \left(p + \tfrac{1}{2} \, \mu_0 \, (H_2^2 + H_3^2) \right) = 0, \tag{3.8}$$

$$\frac{\partial u_j}{\partial t} - \mu_0 \, H_1 \, \frac{\partial H_j}{\partial x} = 0, \quad j = 2, 3$$

$$\frac{\partial S}{\partial t} = 0,$$

Here the fluid quantities are the density ρ, specific volume $\tau = \rho^{-1}$, velocity field (u_1, u_2, u_3), entropy S and hydrostatic pressure $p = p(\tau, S)$, while (H_1, H_2, H_3) is the magnetic field and μ_0 is the magnetic constant [17]. In one-dimensional MHD, the first component H_1 of the magnetic field is necessarily constant.

System (3.8) simplifies significantly if we take the constant H_1 to vanish, so the magnetic field is perpendicular to the direction of motion. In this case, the full system (3.8) can be written

$$\frac{\partial \tau}{\partial t} - \frac{\partial u_1}{\partial x} = 0,$$

$$\frac{\partial u_1}{\partial t} + \frac{\partial \widetilde{p}}{\partial x} = 0, \tag{3.9}$$

regarded as a generalized p-system, and coupled with

$$\frac{\partial \widetilde{H}_j}{\partial t} = 0, \quad \frac{\partial u_j}{\partial t} = 0, \quad \frac{\partial S}{\partial t} = 0, \quad j = 1, 2, \tag{3.10}$$

where we have written

$$\widetilde{H}_j := \tau \, H_j \quad \text{and} \quad \widetilde{p} := p + \frac{1}{2} \, \mu_0 \, \frac{\widetilde{H}_2^2 + \widetilde{H}_3^2}{\tau^2}.$$

System (3.9) is clearly an example of (3.1), so all results in [4] apply directly. Here we obtain a stronger singularity formation result than Lemma 3.2 by restricting our consideration to a polytropic ideal gas,

$$p = K \, e^{\frac{S}{c\tau}} \, \tau^{-\gamma},$$

with adiabatic gas constant $1 < \gamma \le 2$.

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Theorem 3.3. *For system (3.9), (3.10), assume that the initial data is C^2 with uniform bounds*

$$|G(\cdot, 0)| < N_0, \quad |G(\cdot, 0)_x| < N_1, \quad and \quad |G(\cdot, 0)_{xx}| < N_2$$

for each of $G = S$, H_2, H_3. Assume also that the density is globally bounded,

$$0 < N_{\rho L} < \rho(x, t) < N_{\rho U}, \quad for \ any \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \tag{3.11}$$

for constants $N_{\rho L}$ and $N_{\rho U}$. There exists a positive constant $\widehat{N}$, depending on N_j and $N_{\rho U}$, but not on $N_{\rho L}$, such that, if

$$\inf \{\ y(\cdot, 0), \ q(\cdot, 0)\ \} < -\widehat{N},$$

then u_x and/or τ_x blow up in finite time. Here y and q are defined in (3.4), with $P = \widetilde{p}$.

Although we assume the global bounds (3.11), $\widehat{N}$ is independent of the lower bound $N_{\rho L}$ of density. Theorem 2.1 for the compressible Euler equations indicates that it is reasonable to expect an *a priori* upper bound on the density, and such a bound would imply that $\widehat{N}$ depends only on the initial data.

Proof. As in the proof of Theorem 2.4, from (3.6), it suffices to find uniform bounds for the roots of the quadratic equations

$$a_0 + a_1\, y + a_2\, y^2 = 0 \quad and \quad a_0 - a_1\, q + a_2\, q^2 = 0,$$

whose coefficients are given in (3.7) with P replaced by $\widetilde{p}$. That is, we must choose $\widehat{N}$ so that

$$\left|\frac{a_1}{2\, a_2}\right| + \sqrt{\left|\frac{a_1}{2\, a_2}\right|^2 + \frac{a_0}{|a_2|}} \leq \widehat{N}, \tag{3.12}$$

when $a_1^2 - 4a_0 a_2 \geq 0$, where we calculate

$$\left|\frac{a_1}{2\, a_2}\right| = \left|\frac{c\sqrt{c}}{c_h}\left(\frac{\widetilde{p}_\mu}{c}\right)_h + I\right|,$$

and

$$\left|\frac{a_1}{2\, a_2}\right|^2 + \frac{a_0}{|a_2|} = \left(\frac{c\sqrt{c}}{c_h}\left(\frac{\widetilde{p}_\mu}{c}\right)_h\right)^2 - 2\,\frac{c\sqrt{c}}{c_h}\,I_\mu + \frac{c}{c_h}\left(\frac{\widetilde{p}_\mu}{c}\right)_h\,\widetilde{p}_\mu.$$

It follows that we can choose $\widehat{N}$ provided we find upper bounds (independent of $N_{\rho L}$) for each of the quantities

$$\left| \left(\frac{\widetilde{p}_\mu}{c} \right)_h \right|, \quad \left| \frac{\widetilde{p}_\mu}{\sqrt{c}} \right|, \quad \left| \frac{c\sqrt{c}}{c_h} \right|, \quad |I|, \quad \text{and} \quad |I_\mu|; \tag{3.13}$$

we treat each of these terms separately.

In order to simplify the calculation, we denote

$$A(x) := K\, e^{\frac{S}{c\tau}}, \quad B(x) := \frac{1}{2}\, \mu_0\, (\widetilde{H}_2^2 + \widetilde{H}_3^2),$$

so that $A(x)$ and $B(x)$ are positive bounded functions, and $A(x)$ is bounded from below by a positive constant. With this notation,

$$\widetilde{p}(\tau, x) = A(x)\, \tau^{-\gamma} + B(x)\, \tau^{-2},$$

the wave speed is

$$c = \sqrt{-\widetilde{p}_\tau} = \sqrt{\gamma\, A\, \tau^{-\gamma-1} + 2\, B\, \tau^{-3}}, \tag{3.14}$$

and

$$h = \int_\tau^\infty c\, d\tau = \int_\tau^\infty \sqrt{\gamma\, A\, \tau^{-\gamma-1} + 2\, B\, \tau^{-3}}\, d\tau.$$

It follows by (3.11) that $\widetilde{p}$, c and h are bounded above and below. We write

$$0 < N_{hL} < h < N_{hU},$$

where N_{hU} does not depend on $N_{\rho L}$.

1. We first consider $(\frac{\widetilde{p}_\mu}{c})_h$. By (3.2) we have $\widetilde{p}_{\mu h} = c_\mu$, and by (3.3),

$$c\left(\frac{\widetilde{p}_\mu}{c} \right)_h = c_\mu - c_h\, \frac{\widetilde{p}_\mu}{c} = c_x + \frac{c_\tau}{c^2}\, \widetilde{p}_x = \frac{c}{2}\left(\frac{\widetilde{p}_x}{\widetilde{p}_\tau} \right)_\tau,$$

and thus

$$\left(\frac{\widetilde{p}_\mu}{c} \right)_h = \frac{1}{2}\left(\frac{\widetilde{p}_x}{\widetilde{p}_\tau} \right)_\tau = \frac{\widetilde{p}_{x\tau}\, \widetilde{p}_\tau - \widetilde{p}_x\, \widetilde{p}_{\tau\tau}}{2\widetilde{p}_\tau^2}.$$

Now, since $1 < \gamma \le 2$, we have

$$|\widetilde{p}_\tau| = \gamma\, A\, \tau^{-\gamma-1} + 2\, B\, \tau^{-3} \ge \gamma\, A\, \tau^{-\gamma-1},$$

and since A is bounded from below, $(\frac{\widetilde{p}_\mu}{c})_h$ is bounded provided

$$F_1 = (\widetilde{p}_{x\tau}\, \widetilde{p}_\tau - \widetilde{p}_x\, \widetilde{p}_{\tau\tau})\, \tau^{2\gamma+2}$$

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is bounded. It is routine to compute

$$F_1 = (\gamma A_x + 2B_x \tau^{\gamma-2})(\gamma A + 2B\tau^{\gamma-2}) - (A_x + B_x \tau^{\gamma-2})(\gamma(\gamma+1)A + 6B\tau^{\gamma-2}),$$

which is bounded independent of $N_{\rho L}$ for $\gamma \leq 2$.

2. By (3.3),

$$\frac{\widetilde{p}_\mu}{\sqrt{c}} = \frac{\widetilde{p}_x}{\sqrt{c}} + h_x \frac{\widetilde{p}_\tau}{c\sqrt{c}} = \frac{\widetilde{p}_x}{\sqrt{c}} - \sqrt{c}\, h_x,$$

and this is bounded by a constant independent of $N_{\rho L}$, since

$$\frac{\widetilde{p}_x}{\sqrt{c}} = \frac{A_x \tau^{-\gamma} + B_x \tau^{-2}}{(\gamma A \tau^{-\gamma-1} + 2 B \tau^{-3})^{\frac{1}{4}}} = O(1)\, \tau^{-\frac{3\gamma-1}{4}},$$

and

$$h_x = \int_\tau^\infty \frac{\gamma A_x \tau^{-\frac{\gamma+1}{2}} + 2B_x \tau^{\frac{\gamma-5}{2}}}{2\sqrt{\gamma A + 2B\,\tau^{\gamma-2}}}\, d\tau = O(1)\, \tau^{-\frac{\gamma-1}{2}},$$

so $\sqrt{c}\, h_x = O(1)\, \tau^{-\frac{3\gamma-1}{4}}$, where $O(1)$ is a bound depending on x and the lower bound for τ, and thus is independent of $N_{\rho L}$.

3. We similarly calculate

$$\frac{c\sqrt{c}}{c_h} = \frac{c\sqrt{c}}{c_\tau \tau_h} = -\frac{2\, c^{\frac{7}{2}}}{2\, c\, c_\tau} = \frac{2\,(\gamma A\tau^{-\gamma-1} + 2B\tau^{-3})^{\frac{7}{4}}}{\gamma(\gamma+1)A\tau^{-\gamma-2} + 6B\tau^{-4}} = O(1)\, \tau^{-\frac{3\gamma-1}{4}},$$

which is again bounded independent of $N_{\rho L}$.

4. Since the upper bounds of c, h and $|(\frac{\widetilde{p}_\mu}{c})_h|$ don't depend on $N_{\rho L}$, the absolute value of I is bounded above by a constant independent of $N_{\rho L}$.

5. By (3.5), (3.3),

$$I_\mu = \int_{h_0}^h \left(\frac{1}{2}\sqrt{c}\left(\frac{\widetilde{p}_\mu}{c}\right)_h\right)_\mu dh = \int_{h_0}^h \left[\frac{c_\mu}{4\sqrt{c}}\left(\frac{\widetilde{p}_\mu}{c}\right)_h + \frac{\sqrt{c}}{2}\left(\frac{\widetilde{p}_\mu}{c}\right)_{h\mu}\right] dh,$$

so it suffices to bound $\frac{c_\mu}{\sqrt{c}}$ and $\sqrt{c}(\frac{\widetilde{p}_\mu}{c})_{h\mu}$. By (3.3),

$$\left|\frac{c_\mu}{\sqrt{c}}\right| \leq \left|\frac{(c^2)_\tau\, h_x}{2c^2\sqrt{c}}\right| + \left|\frac{(c^2)_x}{2c\sqrt{c}}\right| \leq O(1)\, \tau^{-\frac{\gamma+1}{4}},$$

as above. Finally, estimating as above, we obtain

$$\sqrt{c}\left(\frac{\widetilde{p}_\mu}{c}\right)_{h\mu} = O(1)\, \tau^{-\frac{\gamma+1}{4}},$$

as required.

Since all terms in (3.13) are uniformly bounded, it follows that we can choose $\widehat{N}$ so that (3.12) holds. Moreover, $\widehat{N}$ is independent of $N_{\rho L}$. The remainder of the proof follows that of Theorem 2.4. $\qquad\square$

4 Compressible flow in a variable area duct

4.1 Equations and coordinates

In this section, we first consider inviscid compressible flow in a duct of varying cross section $a(x')$. In Eulerian (spatial) coordinates (x', t'), this flow is described by

$$a_{t'} = 0,$$
$$(a\,\rho)_{t'} + (a\,\rho\,u)_{x'} = 0,$$
$$(a\,\rho\,u)_{t'} + (a\,\rho\,u^2)_{x'} + a\,p_{x'} = 0, \tag{4.1}$$
$$(a\,\rho\,E)_{t'} + (a\,\rho\,E\,u + a\,p\,u)_{x'} = 0,$$

$a = a(x')$ is the cross-sectional area of the duct and $E = e + \frac{1}{2}\,u^2$ is the total energy [6, 7, 12]. The others are the standard thermodynamic variables, subject to (2.4) as usual. The first equation describes the fixed duct, while the others represent conservation of mass, momentum and energy, respectively. For C^1 solutions, the energy equation is equivalent to an entropy equation,

$$S_{t'} + u\,S_{x'} = 0.$$

We use Lagrangian (material) coordinates (x, t), defined by

$$x = \int a(x')\,\rho(x')\,dx', \quad t = t',$$

so that

$$dx = a\,\rho\,dx' - a\,\rho\,u\,dt', \quad \text{and} \quad dt = dt'. \tag{4.2}$$

We define a specific length by

$$v := \frac{1}{a\rho}, \tag{4.3}$$

and rewrite system (4.1) in the Lagrangian frame, to get

$$v_t - u_x = 0,$$
$$u_t + a\,p_x = 0, \tag{4.4}$$
$$S_t = 0,$$

valid for C^1 solutions, which is a variant of the generalized p-system (3.1).

We restrict our attention to a polytropic ideal gas (2.6), so that

$$p = p(v, a, S) = K\,e^{\frac{S}{c_\tau}}\,a^{-\gamma}\,v^{-\gamma},$$

and define the generalized Lagrangian wavespeed by

$$C = \sqrt{-a\, p_v} = \sqrt{K\gamma}\, a^{-\frac{\gamma-1}{2}}\, v^{-\frac{\gamma+1}{2}}\, e^{\frac{S}{2c_\tau}}.$$

As above, we define new variables m and z for S and v by

$$m = e^{\frac{S}{2c_\tau}} \quad \text{and} \quad z = \int_v^\infty \frac{C\, a^{\frac{\gamma-1}{2}}}{m}\, dv = \frac{2\sqrt{K\gamma}}{\gamma-1}\, v^{-\frac{\gamma-1}{2}}.$$

It follows that

$$v = K_\tau\, z^{-\frac{2}{\gamma-1}}, \qquad p = K_p\, a^{-\gamma}\, m^2\, z^{\frac{2\gamma}{\gamma-1}},$$

$$\text{and} \quad C = C(m,z,a) = K_c\, a^{-\frac{\gamma-1}{2}}\, m\, z^{\frac{\gamma+1}{\gamma-1}}, \tag{4.5}$$

where the constants K are given by (2.11). In these coordinates, C^1 solutions of (4.4) satisfy the equivalent system

$$z_t + \frac{C}{m}\, a^{\frac{\gamma-1}{2}}\, u_x = 0,$$

$$u_t + m\, C\, a^{-\frac{\gamma-1}{2}}\, z_x + 2\,\frac{a\, p}{m}\, m_x - \gamma\, p\, a_x = 0, \tag{4.6}$$

$$m_t = 0.$$

4.2 Coupled Riccati equations

Following [2], we define gradient variables by

$$\alpha = u_x + a^{-\frac{\gamma-1}{2}}\, m\, z_x + \frac{\gamma-1}{\gamma}\, a^{-\frac{\gamma-1}{2}}\, m_x\, z,$$

$$\beta = u_x - a^{-\frac{\gamma-1}{2}}\, m\, z_x - \frac{\gamma-1}{\gamma}\, a^{-\frac{\gamma-1}{2}}\, m_x\, z, \tag{4.7}$$

and denote derivatives along forward and backward characteristics by

$$\partial_+ = \partial_t + C\, \partial_x \quad \text{and} \quad \partial_- = \partial_t - C\, \partial_x,$$

respectively.

Although the area $a(x')$ of the duct is stationary in spatial coordinates, it is moving in material coordinates. We write

$$\dot{a} := \frac{da(x')}{dx'} \quad \text{and} \quad \ddot{a} := \frac{d^2 a(x')}{dx'^2},$$

so that, according to (4.2), $a(x,t)$ satisfies

$$a_t = u(x,t)\, \dot{a}(x'), \quad a_x = v(x,t)\, \dot{a}(x') \quad \text{and} \quad a_{xx} = v^2\, \ddot{a} + v_x\, \dot{a}, \tag{4.8}$$

for $x' = x'(x,t)$.

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Lemma 4.1. *For C^1 solutions of (4.1), we have*

$$\partial_+\alpha = k_1 \left(k_2 \left(3\alpha + \beta\right) + \left(\alpha\,\beta - \alpha^2\right)\right) + k_3^+ \left(\alpha - \beta\right) + F(x,t),$$
$$\partial_-\beta = k_1 \left(-k_2 \left(\alpha + 3\beta\right) + \left(\alpha\,\beta - \beta^2\right)\right) + k_3^- \left(\beta - \alpha\right) + F(x,t), \qquad (4.9)$$

where

$$k_1 = \tfrac{\gamma+1}{2(\gamma-1)}\, K_c\, z^{\frac{2}{\gamma-1}}, \qquad k_2 = \tfrac{\gamma-1}{\gamma(\gamma+1)}\, m_x\, z\, a^{-\frac{\gamma-1}{2}},$$
$$k_3^{\pm} = -\tfrac{\gamma-1}{4}\, u\, a^{-1}\dot{a} \pm \tfrac{3(\gamma-1)^2}{8}\, m\, z\, a^{-\frac{\gamma+1}{2}}\,\dot{a},$$

and where

$$F(x,t) = \tfrac{(\gamma-1)^3}{8K_c}\, m^2\, z^{\frac{2\gamma-4}{\gamma-1}}\, a^{-\gamma-1}\left(a\,\ddot{a} - \gamma\,\dot{a}^2\right) + \tfrac{(\gamma-1)^2}{2\gamma}\, m\, m_x\, z^2\, a^{-\gamma}\,\dot{a}.$$

We note that the coefficients k_j and zero-order terms F depend only on the solution itself and on stationary derivatives of initial entropy and duct shape. It is reasonable to assume that $|\dot{a}|$, $|\ddot{a}|$ and $|m_x|$ are uniformly bounded, so the coefficients and inhomogeneity are essentially zero order terms. As in all of the simpler examples, the quadratic nonlinearity drives the growth of α and/or β to infinity, at which time derivatives blow up and shocks form in the solution.

Proof. We derive the equation for α. Differentiating (4.7), we get

$$\partial_+\alpha = \left[u_{xt} + C\,(a^{-\frac{\gamma-1}{2}}\, m\, z_x)_x\right] + \left[(a^{-\frac{\gamma-1}{2}}\, m\, z_x)_t + C\, u_{xx}\right]$$
$$+ \tfrac{\gamma-1}{\gamma}\left[(a^{-\frac{\gamma-1}{2}}\, m_x\, z)_t + C\,(a^{-\frac{\gamma-1}{2}}\, m_x\, z)_x\right].$$

Now differentiate (4.6) in x to get

$$(a^{-\frac{\gamma-1}{2}}\, m\, z_x)_t + C\, u_{xx} = -C_x\, u_x - (a^{-\frac{\gamma-1}{2}}\, m)_x\, z_t + (a^{-\frac{\gamma-1}{2}})_t\, m\, z_x,$$
$$u_{xt} + C\,(m\, a^{-\frac{\gamma-1}{2}}\, z_x)_x = -(2\,\tfrac{a\, p}{m}\, m_x)_x + (\gamma\, p\, a_x)_x - C_x\, m\, a^{-\frac{\gamma-1}{2}}\, z_x,$$

and substitute in to get

$$\partial_+\alpha = -(2\,\tfrac{a\, p}{m}\, m_x)_x + (\gamma\, p\, a_x)_x - C_x\, m\, a^{-\frac{\gamma-1}{2}}\, z_x$$
$$- C_x\, u_x - (a^{-\frac{\gamma-1}{2}}\, m)_x\, z_t + (a^{-\frac{\gamma-1}{2}})_t\, m\, z_x \qquad (4.10)$$
$$+ \tfrac{\gamma-1}{\gamma}\left[(a^{-\frac{\gamma-1}{2}}\, m_x\, z)_t + C\,(a^{-\frac{\gamma-1}{2}}\, m_x\, z)_x\right].$$

Note that we have eliminated all second derivatives except for a_{xx} and m_{xx}, since $m_{xt} = 0$, and we use (4.8) to eliminate a_t, a_x and a_{xx}. The quadratic terms in (4.9) are those involving C_x.

Finally, we express the derivatives u_x, z_t and z_x in terms of α and β. By (4.7), (4.6) and (4.5), we have

$$u_x = \tfrac{1}{2}\,(\alpha + \beta),$$

$$z_x = \frac{a^{\frac{\gamma-1}{2}}}{2\,m}\,(\alpha - \beta) - \frac{\gamma-1}{\gamma\,m}\,z\,m_x,$$

$$z_t = -\frac{K_c}{2}\,z^{\frac{\gamma+1}{\gamma-1}}\,(\alpha + \beta).$$

Substituting these into (4.10) and simplifying yields (4.9). We omit the details, which are similar to those appearing in [2]. $\qquad\square$

4.3 Decoupled Riccati equations

As in [2, 4], equations (4.9) can be rewritten as a decoupled system with varying coefficients. Since the α equation has terms linear in β, these can be eliminated by an integrating factor. By (4.5), (4.6) and (4.7),

$$\partial_+ z = z_t + C\,z_x = -K_c\,z^{\frac{\gamma+1}{\gamma-1}}\,(\beta + \tfrac{\gamma-1}{\gamma}\,m_x\,z\,a^{-\frac{\gamma-1}{2}}),$$

so that

$$\beta = -\frac{1}{K_c}\,z^{-\frac{\gamma+1}{\gamma-1}}\,\partial_+ z - \tfrac{\gamma-1}{\gamma}\,m_x\,z\,a^{-\frac{\gamma-1}{2}}.$$

We substitute for β in (4.9), move terms including $\partial_+ z$ to the left hand side, and multiply by $z^{\frac{\gamma+1}{2(\gamma-1)}}$, to get

$$\partial_+\big(z^{\frac{\gamma+1}{2(\gamma-1)}}\,\alpha\big) + \tfrac{1}{2\gamma}m_x a^{-\frac{\gamma-1}{2}} z^{\frac{\gamma+1}{2(\gamma-1)}}\,\partial_+ z - \frac{k_3^+}{K_c}z^{-\frac{\gamma+1}{2(\gamma-1)}}\,\partial_+ z$$

$$= k_1\big(k_2(3\alpha - \tfrac{\gamma-1}{\gamma}m_x za^{-\frac{\gamma-1}{2}}) + (-\tfrac{\gamma-1}{\gamma}m_x za^{-\frac{\gamma-1}{2}}\alpha - \alpha^2)\big)z^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$+ k_3^+(\alpha + \tfrac{\gamma-1}{\gamma}m_x za^{-\frac{\gamma-1}{2}})z^{\frac{\gamma+1}{2(\gamma-1)}} + F(x,t)z^{\frac{\gamma+1}{2(\gamma-1)}}. \qquad (4.11)$$

Lemma 4.2. *There are variables Y and Q, defined by*

$$Y = z^{\frac{\gamma+1}{2(\gamma-1)}}\,\alpha + \widetilde{Y}(z, u, m, m_x, a, \dot{a}),$$

$$Q = z^{\frac{\gamma+1}{2(\gamma-1)}}\,\beta + \widetilde{Q}(z, u, m, m_x, a, \dot{a}),$$

which satisfy the equations

$$\partial_+ Y = d_0 + d_1\,Y + d_2\,Y^2,$$

$$\partial_- Q = \bar{d}_0 - \bar{d}_1\,Q + d_2\,Q^2, \qquad (4.12)$$

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with coefficients

$$d_2 = -\frac{\gamma+1}{\gamma-1} K_c z^{\frac{3-\gamma}{2(\gamma-1)}},$$

$$d_1, \ \bar{d}_1 = d_1, \ \bar{d}_1(z, u, m, m_x, a, \dot{a}), \quad and$$

$$d_0, \ \bar{d}_0 = d_0, \ \bar{d}_0(z, u, m, m_x, m_{xx}, a, \dot{a}, \ddot{a}).$$

Proof. For convenience, we take $\gamma \neq 5/3$ or 3. Writing (4.11) as a derivative along the characteristic, we get

$$\partial_+ \big(z^{\frac{\gamma+1}{2(\gamma-1)}} \alpha\big) + \partial_+ \big(\frac{\gamma-1}{\gamma(3\gamma-1)} m_x a^{-\frac{\gamma-1}{2}} z^{\frac{3\gamma-1}{2(\gamma-1)}}\big)$$

$$- \frac{3(\gamma-1)^2}{8K_c} m \dot{a} \, a^{-\frac{\gamma+1}{2}} z^{\frac{\gamma-3}{2(\gamma-1)}} \partial_+ z + \partial_+ \big(\frac{(\gamma-1)^2}{2K_c(\gamma-3)} u \, a^{-1} \dot{a} \, z^{\frac{\gamma-3}{2(\gamma-1)}}\big)$$

$$= \frac{(\gamma-1)^2}{2K_c(\gamma-3)} a^{-1} \dot{a} \, z^{\frac{\gamma-3}{2(\gamma-1)}} \partial_+ u$$

$$+ \partial_+ \big(\frac{\gamma-1}{\gamma(3\gamma-1)} m_x a^{-\frac{\gamma-1}{2}}\big) z^{\frac{3\gamma-1}{2(\gamma-1)}} + \frac{(\gamma-1)^2}{2K_c(\gamma-3)} u \, z^{\frac{\gamma-3}{2(\gamma-1)}} \partial_+ (a^{-1} \dot{a})$$

$$+ k_1 \big(k_2(3\alpha - \frac{\gamma-1}{\gamma} m_x z a^{-\frac{\gamma-1}{2}}) + (-\frac{\gamma-1}{\gamma} m_x z a^{-\frac{\gamma-1}{2}} \alpha - \alpha^2)\big) z^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$+ k_3^+ (\alpha + \frac{\gamma-1}{\gamma} m_x z a^{-\frac{\gamma-1}{2}}) z^{\frac{\gamma+1}{2(\gamma-1)}} + F(x,t) z^{\frac{\gamma+1}{2(\gamma-1)}}.$$

Next, we eliminate the $\partial_+ u$ term, which comes from differentiating k_3^+. By (4.6), we have

$$\partial_+ u = u_t + C u_x = -m a^{-\frac{\gamma-1}{2}} \partial_+ z - \frac{\gamma-1}{\gamma} C \, z \, m_x \, a^{-\frac{\gamma-1}{2}} + \gamma \, p \, a_x,$$

and we again write this as a total derivative plus zero-order terms in z and u.

Finally, we define

$$Y = z^{\frac{\gamma+1}{2(\gamma-1)}} \alpha + \frac{(\gamma-1)^2}{2K_c(\gamma-3)} u \, a^{-1} \dot{a} \, z^{\frac{\gamma-3}{2(\gamma-1)}} + \frac{\gamma-1}{\gamma(3\gamma-1)} m_x \, a^{-\frac{\gamma-1}{2}} z^{\frac{3\gamma-1}{2(\gamma-1)}}$$

$$- \frac{(3\gamma-13)(\gamma-1)^3}{4K_c(\gamma-3)(3\gamma-5)} m \dot{a} \, a^{-\frac{\gamma+1}{2}} z^{\frac{3\gamma-5}{2(\gamma-1)}}, \tag{4.13}$$

and simplify, to get

$$\partial_+ Y = -z^{\frac{3\gamma-5}{2(\gamma-1)}} \partial_+ \left(\frac{(3\gamma-13)(\gamma-1)^3}{4K_c(\gamma-3)(3\gamma-5)} m\,\dot{a}\,a^{-\frac{\gamma+1}{2}} \right)$$

$$+ \frac{(\gamma-1)^2}{2K_c(\gamma-3)}\, a^{-1}\,\dot{a}\, z^{\frac{\gamma-3}{2(\gamma-1)}} \left(-\frac{\gamma-1}{\gamma} C\, z\, m_x\, a^{-\frac{\gamma-1}{2}} + \gamma\, p\, a_x \right)$$

$$+ \left(\frac{\gamma-1}{\gamma(3\gamma-1)}\, m_x\, a^{-\frac{\gamma-1}{2}} \right) z^{\frac{3\gamma-1}{2(\gamma-1)}} + \frac{(\gamma-1)^2}{2K_c(\gamma-3)}\, u\, z^{\frac{\gamma-3}{2(\gamma-1)}} \partial_+(a^{-1}\dot{a})$$

$$+ k_1 \left(k_2 \left(3\alpha - \frac{\gamma-1}{\gamma} m_x z a^{-\frac{\gamma-1}{2}} \right) + \left(-\frac{\gamma-1}{\gamma} m_x z a^{-\frac{\gamma-1}{2}} \alpha - \alpha^2 \right) \right) z^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$+ k_3^+ \left(\alpha + \frac{\gamma-1}{\gamma} m_x z a^{-\frac{\gamma-1}{2}} \right) z^{\frac{\gamma+1}{2(\gamma-1)}} + F(x,t) z^{\frac{\gamma+1}{2(\gamma-1)}}.$$

Using (4.13) to write

$$Y = z^{\frac{\gamma+1}{2(\gamma-1)}}\, \alpha + y_0, \quad \text{so} \quad \alpha = z^{-\frac{\gamma+1}{2(\gamma-1)}} Y - z^{-\frac{\gamma+1}{2(\gamma-1)}}\, y_0,$$

we eliminate α to get

$$\partial_+ Y = d_0 + d_1\, Y + d_2\, Y^2,$$

with coefficients d_i containing no derivatives of z, u, and in particular,

$$d_2 = -\frac{\gamma+1}{\gamma-1} K_c\, z^{\frac{3-\gamma}{2(\gamma-1)}}.$$

Using $z^{\frac{2}{\gamma-1}} = K_\tau\, a\, \rho$, after a tedious calculation we get, for $\gamma \neq 5/3$ or 3,

$$d_0 = a^{\frac{\gamma-3}{4}} \left(\frac{1}{\gamma} K_c K_\tau^{\frac{5\gamma+1}{4}} \left(\frac{\gamma-1}{3\gamma-1} mm_{xx} - \frac{(3\gamma+1)(\gamma-1)}{(3\gamma-1)^2} m_x^2 \right) a^2\, \rho^{\frac{5\gamma+1}{4}} \right.$$

$$+ L_1\, m_x\, \dot{a}\, u\, \rho^{\frac{3\gamma-1}{4}} + L_2\, m\, m_x\, \dot{a}\, \rho^{\frac{5\gamma-3}{4}}$$

$$+ L_3\, m\, \dot{a}^2\, a^{-2}\, u\, \rho^{\frac{3\gamma-5}{4}} + L_4\, m\, \ddot{a}\, a^{-1}\, u\, \rho^{\frac{3\gamma-5}{4}}$$

$$+ L_5\, m^2\, a^{-2}\, \dot{a}^2\, \rho^{\frac{5\gamma-7}{4}} + L_6\, \ddot{a}\, a^{-1}\, u^2\, \rho^{\frac{\gamma-3}{4}}$$

$$\left. + L_7\, a^{-2}\, \dot{a}^2\, u^2\, \rho^{\frac{\gamma-3}{4}} + L_8\, m^2\, \ddot{a}\, a^{-1}\, \rho^{\frac{5\gamma-7}{4}} \right),$$

$$d_1 = L_9\, m_x\, a\, \rho^{\frac{\gamma+1}{2}} + L_{10}\, u\, a^{-1}\, \dot{a} + L_{11}\, m\, a^{-1}\, \dot{a}\, \rho^{\frac{\gamma-1}{2}},$$

$$d_2 = -\frac{\gamma+1}{\gamma-1} K_c\, K_\tau^{\frac{3-\gamma}{4}}\, a^{\frac{3-\gamma}{4}}\, \rho^{\frac{3-\gamma}{4}} < 0,$$

where the L_j are constants depending only on γ.

Similarly, if Q is defined by

$$Q = z^{\frac{\gamma+1}{2(\gamma-1)}}\, \alpha + \frac{(\gamma-1)^2}{2K_c(\gamma-3)}\, u\, a^{-1}\, \dot{a}\, z^{\frac{\gamma-3}{2(\gamma-1)}} - \frac{\gamma-1}{\gamma(3\gamma-1)}\, m_x\, a^{-\frac{\gamma-1}{2}}\, z^{\frac{3\gamma-1}{2(\gamma-1)}}$$

$$+ \frac{(3\gamma-13)(\gamma-1)^3}{4K_c(\gamma-3)(3\gamma-5)}\, m\, \dot{a}\, a^{-\frac{\gamma+1}{2}}\, z^{\frac{3\gamma-5}{2(\gamma-1)}},$$

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then it satisfies
$$\partial_- Q = \bar{d}_0 + \bar{d}_1 \, Q + d_2 \, Q^2,$$
with, for $\gamma \neq 5/3$ or 3,

$$
\begin{aligned}
\bar{d}_0 = {} & a^{\frac{\gamma-3}{4}} \Big(\tfrac{1}{\gamma} K_c K_\tau^{\frac{5\gamma+1}{4}} \big(\tfrac{\gamma-1}{3\gamma-1} m m_{xx} - \tfrac{(3\gamma+1)(\gamma-1)}{(3\gamma-1)^2} m_x^2 \big) a^2 \, \rho^{\frac{5\gamma+1}{4}} \\
& + \bar{L}_1 \, m_x \, \dot{a} \, u \, \rho^{\frac{3\gamma-1}{4}} + \bar{L}_2 \, m \, m_x \, \dot{a} \, \rho^{\frac{3\gamma-3}{4}} \\
& + \bar{L}_3 \, m \, \dot{a}^2 \, a^{-2} \, u \, \rho^{\frac{3\gamma-5}{4}} + \bar{L}_4 \, m \, \ddot{a} \, a^{-1} \, u \, \rho^{\frac{3\gamma-5}{4}} \\
& + \bar{L}_5 \, m^2 \, a^{-2} \, \dot{a}^2 \, \rho^{\frac{5\gamma-7}{4}} + \bar{L}_6 \, \ddot{a} \, a^{-1} \, u^2 \, \rho^{\frac{\gamma-3}{4}} \\
& + \bar{L}_7 \, a^{-2} \, \dot{a}^2 \, u^2 \, \rho^{\frac{\gamma-3}{4}} + \bar{L}_8 \, m^2 \, \ddot{a} \, a^{-1} \, \rho^{\frac{5\gamma-7}{4}} \Big), \\
\bar{d}_1 = {} & \bar{L}_9 \, m_x \, a \, \rho^{\frac{\gamma+1}{2}} + \bar{L}_{10} \, u \, a^{-1} \, \dot{a} + \bar{L}_{11} \, m \, a^{-1} \, \dot{a} \, \rho^{\frac{\gamma-1}{2}},
\end{aligned}
$$

and $\bar{L}_j$ are constants depending only on γ. It is clear that similar equations hold for $\gamma = 5/3$ or 3. $\qquad\square$

4.4 Singularity formation

Having obtained Riccati equations (4.12) for derivative variables, we obtain a gradient blowup result as in Theorem 2.4. The result states that a strong enough compressive wave will eventually form a shock, provided the state variables remain finite. Theorem 2.1 indicates that state variables should indeed remain bounded for all times, although a vacuum can form in infinite time [26, 5].

Corollary 4.3. *Assume there exist some positive constants M_i, such that the duct and initial data satisfy*

$$
\begin{aligned}
M_1 < a(x,0) < M_2, \quad |\dot{a}(x,0)| < M_3, \quad |\ddot{a}(x,0)| < M_4, \\
M_5 < m(x,0) < M_6, \quad |m_x(x,0)| < M_7, \quad |m_{xx}(x,0)| < M_8,
\end{aligned}
$$

while the solutions satisfy the a priori *bounds*

$$|u(x,t)| < M_9, \quad \rho(x,t) < M_{10},$$

and $\rho(x,t) > M_{11}$ if $1 < \gamma \leq 3$, for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Then there exists a positive constant $\bar{N}$, depending on γ and M_i, such that, if

$$Y < -\bar{N} \quad or \quad Q < -\bar{N}$$

somewhere in the initial data, then a singularity forms in finite time.

Proof. The roots of (4.12), if they exist, are given by the quadratic formula

$$\frac{-d_1 \pm \sqrt{d_1^2 - 4\,d_0\,d_2}}{2\,d_2} \quad \text{or} \quad \frac{\bar{d}_1 \pm \sqrt{\bar{d}_1^2 - 4\,\bar{d}_0\,d_2}}{2\,d_2},$$

which have bounds depending on the ratios $|d_1|/|d_2|$, $|d_0|/|d_2|$, $|\bar{d}_1|/|d_2|$ and $|\bar{d}_0|/|d_2|$. It follows that these roots have a lower bound $-\bar{N}$ depending on the M_i, and the proof follows in the same way as Theorem 2.4. $\qquad \Box$

4.5   Spherically symmetric Euler equations

The equations for compressible flow in a duct include spherically symmetric three-dimensional Euler flow as a special case, on the exterior of a ball. Using $r = |x'|$ as the radial coordinate, the Euler equations reduce to the duct equations with x replaced by r and $a(r) = r^2$.

In order to get cleaner equations, we consider the isentropic case, with $m = 1$, say, and $\gamma \neq 5/3$ or 3. With these assumptions, $m_x = 0$, $\dot{a} = 2\,r$ and $\ddot{a} = 2$, and (4.12) hold with coefficients

$$d_0 = r^{\frac{\gamma-3}{2}-2}\,\rho^{\frac{\gamma-3}{4}}\big(G_1\,u\,\rho^{\frac{\gamma-1}{2}} + G_2\,\rho^{\gamma-1} + G_3\,u^2\big)$$
$$d_1 = r^{-1}\big(G_4\,u + G_5\,\rho^{\frac{\gamma-1}{2}}\big)$$
$$d_2 = -\tfrac{\gamma+1}{\gamma-1}K_c\,K_\tau^{\frac{3-\gamma}{4}}\,r^{\frac{3-\gamma}{2}}\,\rho^{\frac{3-\gamma}{4}},$$

for constants G_i. Thus the ratios are

$$\frac{d_1}{d_2} = r^{\frac{\gamma-5}{2}}\,\rho^{\frac{\gamma-3}{4}}\big(\hat{G}_4\,u + \hat{G}_5\,\rho^{\frac{\gamma-1}{2}}\big)$$
$$\frac{d_0}{d_2} = r^{\gamma-5}\,\rho^{\frac{\gamma-3}{2}}\big(\hat{G}_1\,u\,\rho^{\frac{\gamma-1}{2}} + \hat{G}_2\,\rho^{\gamma-1} + \hat{G}_3\,u^2\big),$$

for constants $\hat{G}_i$. It follows that the roots $\beta_\pm = \frac{-d_1 \pm \sqrt{d_1^2 - 4 d_0 d_2}}{2 d_2}$ of (4.12) satisfy the bound

$$\beta_\pm \geq -r^{\frac{\gamma-5}{2}}\,\rho^{\frac{\gamma-3}{4}}\big(\hat{G}_6\,|u| + \hat{G}_7\,\rho^{\frac{\gamma-1}{2}}\big),$$

for constants $\hat{G}_j$ depending only on the equation of state.

A gradient blowup result similar to Theorem 2.4 now follows exactly as before for the *exterior* problem when $1 < \gamma < 5$.

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