Singularity formation for compressible Euler equations

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Abstract

In this paper, for the p-system and full compressible Euler equations in one space dimension, we provide an equivalent and a sharp condition on initial data, respectively, under which the classical solution must break down in finite time. Moreover, we provide time-dependent lower bounds on density for arbitrary classical solutions for these two equations. Our results have no restriction on the size of solutions.

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1 Introduction

In this paper, we consider the initial value problem for the compressible Euler equations in Lagrangian coordinates in one space dimension,

$$\tau_t - u_x = 0\,, \tag{1.1}$$

$$u_t + p_x = 0\,, \tag{1.2}$$

$$\Big(\frac{1}{2}u^2 + e\Big)_t + (u\,p)_x = 0\,, \tag{1.3}$$

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where ρ is the density, $\tau = \rho^{-1}$ is the specific volume, p is the pressure, u is the velocity, e is the specific internal energy, $t \in \mathbb{R}^+$ is the time and $x \in \mathbb{R}$ is the spatial coordinate. This model is used to describe the gas dynamics. We assume that the gas is ideal polytropic, so

$$p = K\, e^{\frac{S}{c_v}}\, \tau^{-\gamma} \quad \text{with adiabatic gas constant} \quad \gamma > 1\,, \tag{1.4}$$

and

$$e = \frac{p\tau}{\gamma - 1}\,,$$

where S is the entropy, K and c_v are positive constants, see [7] or [24]. For most gases, $1 < \gamma < 3$.

For C^1 solutions, it follows that (1.3) is equivalent to the "entropy equation":

$$S_t = 0\,. \tag{1.5}$$

When the entropy is constant, the flow is isentropic, then (1.1) and (1.2) become a closed system, known as the p-system:

$$\tau_t - u_x = 0\,, \tag{1.6}$$

$$u_t + p_x = 0\,, \tag{1.7}$$

with

$$p = K\, \tau^{-\gamma}\,, \tag{1.8}$$

where, without loss of generality, we still use K to denote the constant in pressure.

Compressible Euler equations and p-system are two of the most important models for hyperbolic conservations laws

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0\,, \tag{1.9}$$

where $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^n$ is the unknown vector and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear flux. System (1.9) typically admits discontinuity, i.e. shock wave, even when initial data are C^∞. The lack of regularity is the major difficulty in analyzing these systems. Now the well-posedness of small total variation solutions for (1.9) including Compressible Euler equations and p-system are fairly well understood [1, 8]. However, large data results, which means results without restriction on the size of solutions, are still very limited.

The main purpose of this paper is to study the breakdown of large data classical solutions for both p-system and full Euler equations, which is related to the formation of shock wave. The previous large data results by

Lax in [10] for p-system and the first author, R. Young and Q. Zhang in [6] for full Euler equations do not include the most practical case $1 < \gamma < 3$ in gas dynamics. After resolving the case when $1 < \gamma < 3$ in this paper, we will give a complete picture on the mechanism of breakdown of classical solutions for both p-system and full Euler equations.

Singularity formation for hyperbolic conservation laws has been studied by a big amount of articles. A survey on the history of these articles can be found in [8]. In one space dimension, the singularity formation for small data solution, i.e. solution around a constant state, has been well understood, where we refer the reader to [9, 10, 12, 13, 18].

For large data problem, scalar conservation law has been fairly completely understood [8]. For uniformly strictly hyperbolic system with two unknowns, Lax provided singularity formation result [10] in 1964. His result can be directly applied to p-system (1.6)∼(1.8) when $\gamma \geq 3$.

However Lax's proof does not cover the p-system when $1 < \gamma < 3$, because the system might lose its uniformly strict hyperbolicity as density goes to zero in infinite time. Indeed, a Riemann problems connecting two extreme sides of two interacting strong rarefaction waves generates vacuum instantaneously when $t > 0$, [24]. Smoothing out this data implies the existence of a C^1-solution such that $\inf_{(x,t)} \rho(x,t) \to 0$ as $t \to +\infty$.

We can also see the difficulty of proving singularity formation from the Riccati equation established by Lax. The p-system satisfies some Riccati equation

$$y' = -ay^2 \,, \tag{1.10}$$

where $y(t)$ denotes some gradient variable, a is a positive function on density and the derivative is along a characteristic direction. To prove the singularity formation when $y(0) < 0$, i.e. initial data have compression, one needs to show that

$$\int_0^\infty a \; dt = \infty \,, \tag{1.11}$$

where the integral is along a characteristic. For uniformly strictly hyperbolic system or small data problem or large data problem of p-system with $\gamma \geq 3$, the leading coefficient in the Riccati equation is uniformly away from zero hence (1.11) is clearly correct. However, for p-system with $\gamma \in (1, 3)$, coefficient a is vanishing as density approaches zero.

In this paper, when $1 < \gamma < 3$, we establish a time-dependent lower bound on density, using which we prove (1.11) and then the singularity formation when initial data have compression together with Lax's decomposition. Combing with existing results when $\gamma \geq 3$, we prove our main theorem for p-system: Theorem 2.4. This theorem can be understood as:

1. *For p-system (1.6)∼(1.8) with smooth initial data and $\gamma > 1$, classical solution breaks down if and only if the initial data are forward or backward compressive somewhere.*

Remark 1.1. *This theorem gives a complete picture of the mechanism of singularity formation for isentropic gas. Here a wave is compression when gradient variable s_x or r_x is negative somewhere, where s and r are some Riemann invariants in forward and backward directions respectively which will be specified later.*

Furthermore, p-system with general pressure law satisfies a similar result, which is given in Theorem 2.9.

Then we consider the non-isentropic Euler equations (1.1)∼(1.4). Variation of entropy makes the extension of large data result from (1.6)∼(1.8) to (1.1)∼(1.4) highly nontrivial. For example, in p-system, Riemann invariants are constant along characteristics, while this is not true for full Euler equations. And Riccati equations for full Euler equations are in a more complicated form comparing to (1.10).

In [6], the first author, R. Young and Q. Zhang first resolved the singularity formation when $\gamma \geq 3$, when initial entropy has finite total variation. In this case, the leading coefficient in the Riccati equation does not vanish when density approaches zero, on the contrary, the leading coefficient vanishes as density approaches infinity. By studying the propagation of Riemann invariants, the authors in [6] established a uniform upper bound on density for smooth solution when initial entropy has finite total variation and then prove the singularity formation result by analyzing the Riccati equations found in [2, 17].

However, in the case when $1 < \gamma < 3$, one meets a more essential difficulty comparing to the case when $\gamma \geq 3$, similar as p-system, related to the loss of uniformly strict hyperbolicity near vacuum. In fact, the leading coefficient in the Riccati equation is also vanishing when density approaches zero. One needs some lower bound estimates on density. In this paper, we establish a time-dependent lower bound on density for any smooth solutions in Corollary 3.7, which we believe is the first one for non-isentropic gas dynamics to the limit of our knowledge. Then we prove singularity formation result for $1 < \gamma < 3$. Our result can be understood as (see precise statement at Theorems 3.5 and 3.10):

2. *For compressible Euler equations (1.1)∼(1.4) with smooth initial data and $\gamma > 1$, classical solution breaks down if the initial data satisfy some condition, describing compression, somewhere.*

Remark 1.2. *In Subsection 3.5, we find smooth stationary solutions without compression showing that the conditions on initial data for singularity formation provided in Theorems 3.5 is a sharp condition, under which classical solution must break down in finite time. Shock-free examples in [5, 17, 25] will also give us some clues why this condition is sharp.*

Furthermore, the result for full Euler equations is consistent with the one for p-system: when the initial entropy oscillation is weak enough, classical solution must break down in finite time even when there are only weak initial compressions.

Our results are also correct for Euler equations in Eulerian coordinates, whose classical solution is equivalent to the one in Lagrangian coordinates, c.f. [8, 24].

Note that the singularities we study are corresponding to shock formation, see Remark 2.6 for an explanation. At the time of the blowup, the L^∞ norm of u, τ, ρ, e and S are all finite. Our conclusions are all large data results, i.e. there are no restriction on the size of the solutions.

Other large data singularity formation results for Euler equations in one space dimension can be found at [4, 5]. Other research works considering the rate on which density approaches zero for specially solution can be found at [14] for p-system and [5] for full system. And there are several papers discussing the difficulty for the well-posedness of Euler equations caused by the vacuum [3, 19].

For compressible Euler equations in multiple space dimensions, there are also some singularity formation results [6, 20, 22, 23]. However, different from the Euler equations in one space dimension, the mechanism of singularity formation in multiple space dimension is still far from fully understood.

This paper is divided into three sections. In Section 2, we review Lax's original result in [10], then prove our singularity formation result for p-system when $1 < \gamma < 3$. In Section 3, we prove our results for the compressible Euler equations.

2 Singularity formation for p-system

In this section, we consider singularity formation for p-system (1.6)∼(1.8).

The proof of our main theorem (Theorem 2.4) is based on the study of Lax's characteristic decomposition established first for general hyperbolic system with two unknowns in [10]. To make this paper self-contained, we first review Lax's decomposition in Subsection 2.1.

Then in Subsection 2.2, we apply Lax's decomposition to p-system with γ-law pressure and prove a singularity formation result when $\gamma \geq 3$ using similar argument as Lax used for general strictly hyperbolic system with two unknowns.

Unfortunately, we could not directly use Lax's argument when $1 < \gamma < 3$, because density might approach zero as time goes to infinity. To overcome this difficulty, we need to find a time-dependent lower bound on density, which will be done in Subsection 2.3. This finally directs to the proof of Theorem 2.4.

Finally, in Subsection 2.4, we extend the result for p-system with γ-law pressure to p-system with general pressure.

2.1 Lax's result for system with two unknowns

This part is basically taken from Lax's paper [10] in 1964. Consider a system of two first-order partial differential equations

$$\begin{aligned} u_t + f_x &= 0\,, \\ v_t + g_x &= 0\,, \end{aligned} \tag{2.1}$$

where f and g are functions of u and v. Carrying out the differentiation in (2.1), we obtain

$$\mathbf{u}_t + A\,\mathbf{u}_x = 0\,, \tag{2.2}$$

where

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}.$$

Suppose that this system is strictly hyperbolic, i.e. the matrix A has real and distinct eigenvalues λ and μ for relevant values of u and v. Use $\mathbf{l}_\lambda$ and $\mathbf{l}_\mu$ to denote the left eigenvectors corresponding to eigenvalues λ and μ, respectively.

Multiplying (2.2) by $\mathbf{l}_\lambda$ and $\mathbf{l}_\mu$ respectively, we have

$$\mathbf{l}_\lambda \cdot \mathbf{u}' = 0\,, \qquad \mathbf{l}_\mu \cdot \mathbf{u}^{\backprime} = 0\,,$$

where we denote

$${}' = \partial_t + \lambda\partial_x\,, \qquad {}^{\backprime} = \partial_t + \mu\partial_x\,.$$

Suppose there exists an integrating factor ϕ such that

$$w' = \phi_\lambda\,\mathbf{l}_\lambda \cdot \mathbf{u}' = 0\,. \tag{2.3}$$

Such ϕ always exists at least locally. Similarly, we have

$$z^{\backprime} = 0$$

for some functions $w(u,v)$ and $z(u,v)$, which are called Riemann invariants along characteristics with characteristic speeds λ and μ, respectively. For general hyperbolic systems with two unknowns, there always exist two Riemann invariants for different families, if we restrict our consideration to the small data solution, i.e. solution which is around a constant state. Furthermore, for p-system, there exist Riemann invariants for both characteristic families for arbitrary solutions.

Then we differentiate $w' = 0$ in (2.3) on x, we have

$$w_{tx} + \lambda w_{xx} + \lambda_w w_x^2 + \lambda_z w_x z_x = 0\,. \tag{2.4}$$

Also by (2.3),

$$0 = z^{\backprime} = z' - (\lambda - \mu) z_x\,,$$

so

$$z_x = \frac{z'}{\lambda - \mu}\,. \tag{2.5}$$

Substitute (2.5) into (2.4) and denote

$$\alpha := w_x\,,$$

then we have

$$\alpha' + \lambda_w \alpha^2 + \frac{\lambda_z}{\lambda - \mu} z' \alpha = 0\,. \tag{2.6}$$

Denote by h a function of w and z which satisfies

$$h_z = \frac{\lambda_z}{\lambda - \mu}\,.$$

Using $w' = 0$ in (2.3), we have

$$h' = h_w w' + h_z z' = \frac{\lambda_z}{\lambda - \mu} z'\,.$$

Substitute this into (2.6) gives

$$\alpha' + \lambda_w \alpha^2 + h' \alpha = 0\,. \tag{2.7}$$

Multiplying (2.7) by e^h and denoting

$$\widetilde{\alpha} := e^h \alpha\,,$$

we have

$$\widetilde{\alpha}' = a(t)\widetilde{\alpha}^2 \tag{2.8}$$

with

$$a(t) := -e^{-h}\lambda_w\,.$$

This Riccati equation gives us a framework for studying the singularity formation and global existence of classical solutions for hyperbolic system with two unknowns. In fact, we could formally solve gradient variable $\widetilde{\alpha}$ along a characteristic with speed λ:

$$\frac{1}{\widetilde{\alpha}(t)} = \frac{1}{\widetilde{\alpha}(0)} + \mathop{\int\!\!\!\!\!\nearrow}\limits_0^t a(t)\;dt$$

where the integral is taken on a characteristic with speed λ.

For simplicity, suppose that $a(t)$ is always non-zero, which is also satisfied by the solution of p-system if initially $a \neq 0$. Without loss of generality, we only consider the case that $a(t) > 0$. If $\widetilde{\alpha}(0) < 0$, i.e. initial solution is compressive somewhere in the λ direction, then $\widetilde{\alpha}(t)$ breaks down under an **extra condition** that

$$\mathop{\int\!\!\!\!\!\nearrow}\limits_0^t a(t)\;dt = \infty\,. \tag{2.9}$$

In Lax's original proof of singularity formation result, he only consider the hyperbolic system with uniformly strict hyperbolicity, i.e. characteristic speeds λ and μ are uniformly away from each other. In this case, $a(t)$ has a positive lower bound hence (2.9) is automatically correct. If we restrict our consideration to small data problems, the function $a(t)$ also has positive lower bound if it initially has one.

However, for large data problem, $a(t)$ does not in general have positive lower bound, even for one of the most important example of (2.1): p-system (1.6)$\sim$(1.8) with $1 < \gamma < 3$. Hence, Lax's result only covers special hyperbolic systems with two unknowns when we consider large data problems, such as p-system (1.6)$\sim$(1.8) with $\gamma \geq 3$.

To resolve this issue for p-system with $1 < \gamma < 3$, in this paper, we establish a time dependent lower bound on $a(t)$ which helps proving (2.9) hence directs to a singularity formation result. Detail is given in Subsection 2.3.

2.2 Application of Lax's result to p-system

We apply Lax's decomposition and singularity formation result to the Cauchy problem of (1.6)∼(1.8) with smooth initial data $u(x,0)$ and $\tau(x,0)$.

We use the following coordinates, c.f. [2]. Denote

$$\eta := \int_\tau^\infty c\,d\tau = \tfrac{2\sqrt{K\gamma}}{\gamma-1}\,\tau^{-\frac{\gamma-1}{2}} > 0\,, \tag{2.10}$$

where the nonlinear Lagrangian sound speed c is

$$c := \sqrt{-p_\tau} = \sqrt{K\,\gamma}\,\tau^{-\frac{\gamma+1}{2}}\,. \tag{2.11}$$

It follows that

$$\begin{aligned} \tau &= K_\tau\,\eta^{-\frac{2}{\gamma-1}}\,,\\ p &= K_p\,\eta^{\frac{2\gamma}{\gamma-1}}\,,\\ c &= \sqrt{-p_\tau} = K_c\,\eta^{\frac{\gamma+1}{\gamma-1}}\,, \end{aligned} \tag{2.12}$$

where K_τ, K_p and K_c are positive constants given by

$$K_\tau := \Big(\frac{2\sqrt{K\gamma}}{\gamma-1}\Big)^{\frac{2}{\gamma-1}}\,,\quad K_p := K\,K_\tau^{-\gamma},\quad\text{and}\quad K_c := \sqrt{K\gamma}\,K_\tau^{-\frac{\gamma+1}{2}}\,, \tag{2.13}$$

so that also

$$K_p = \tfrac{\gamma-1}{2\gamma}K_c \quad\text{and}\quad K_\tau K_c = \tfrac{\gamma-1}{2}\,. \tag{2.14}$$

In this paper, we always use K with some subscripts to denote positive constants. We will not notify the reader again if there is no ambiguity.

The forward and backward characteristics are described by

$$\frac{dx}{dt} = c \quad\text{and}\quad \frac{dx}{dt} = -c\,,$$

and we denote the corresponding directional derivatives along these by

$$\partial_+ := \tfrac{\partial}{\partial t} + c\,\tfrac{\partial}{\partial x} \quad\text{and}\quad \partial_- := \tfrac{\partial}{\partial t} - c\,\tfrac{\partial}{\partial x}\,,$$

respectively. Furthermore, the Riemann invariants are

$$r := u - \eta \quad\text{and}\quad s := u + \eta\,, \tag{2.15}$$

which satisfy

$$\partial_+ s = 0 \quad\text{and}\quad \partial_- r = 0\,, \tag{2.16}$$

respectively.

Then we denote gradient variables

$$y := \eta^{\frac{\gamma+1}{2(\gamma-1)}}\, s_x \quad \text{and} \quad q := \eta^{\frac{\gamma+1}{2(\gamma-1)}}\, r_x \,,$$

and show y and q satisfy following Riccati equations:

Lemma 2.1. *For C^1 solutions of (1.6)-(1.8), we have*

$$\partial_+ y = -a_2\, y^2 \,, \tag{2.17}$$

$$\partial_- q = -a_2\, q^2 \,, \tag{2.18}$$

where

$$a_2 := K_c\, \tfrac{\gamma+1}{2(\gamma-1)}\, \eta^{\frac{3-\gamma}{2(\gamma-1)}} \,. \tag{2.19}$$

This lemma is an easy corollary of Lax's decomposition (2.8), where the detail calculation can be found in [2].

Proposition 2.2. *(A corollary from [10]) Assume that initial data $u(x,0)$ and $\tau(x,0))$ are C^1, $|u(x,0)|$, $\tau(x,0)$, $|s_x(x,0)|$ and $|r_x(x,0)|$ are uniformly bounded above, and $\tau(x,0)$ is uniformly away from zero. When $\gamma \geq 3$, classical solution of (1.6)$\sim$(1.8) breaks down if*

$$s_x(x,0) < 0 \quad \textit{or} \quad r_x(x,0) < 0 \,. \tag{2.20}$$

Proof. We will show that if $s_x(x,0) < 0$ or $r_x(x,0) < 0$ for some x, then singularity forms in finite time. Without loss of generality, we assume that $s_x(x^*,0) < 0$, then $y(x^*,0) < 0$ for some x^*. Then we denote the forward characteristic passing $(x^*,0)$ as $x^+(t)$. By (2.17),

$$\frac{1}{y(x^+(t),t)} = \frac{1}{y(x^*,0)} + \int_0^t a_2\, dt \,, \tag{2.21}$$

where recall

$$a_2 = K_c\, \tfrac{\gamma+1}{2(\gamma-1)}\, \eta^{\frac{3-\gamma}{2(\gamma-1)}} \,.$$

By (2.15) and (2.16), η is uniformly bounded above, so a_2 has a positive constant lower bound when $\gamma \geq 3$, hence right hand side of (2.21) approaches zero in finite time, which means singularity happens in finite time. $\square$

However, when $1 < \gamma < 3$, the function a_2 does not have constant positive lower bound, because the density has no constant positive lower bound. For example, in the interaction between two strong rarefaction simple waves, the density approaches zero as time goes to infinity.

2.3 Breakdown of classical solutions in p-system when $1 < \gamma < 3$

In this section, we prove the singularity formation for the Cauchy problem in p-system when $1 < \gamma < 3$, by providing a time dependent positive lower bound on density.

Before showing the main theorem, we first give a lemma:

Lemma 2.3. *For C^1 solutions of (1.6)-(1.8), we have* a priori *bounds*

$$y(x,t) \le \max\Big\{0,\ \sup_x\{y(x,0)\}\Big\} =: Y$$

$$\textit{and} \quad q(x,t) \le \max\Big\{0,\ \sup_x\{q(x,0)\}\Big\} =: Q\,.$$

Proof. This Lemma is easily proved by Lemma 2.1. □

Then we give the first main theorem in this paper.

Theorem 2.4. *Assume that initial data $(u(x,0),\tau(x,0))$ are C^1, $|u(x,0)|$, $\tau(x,0)$, $|s_x(x,0)|$ and $|r_x(x,0)|$ are uniformly bounded above, and $\tau(x,0)$ is uniformly away from zero. Global-in-time classical solution of (1.6)∼(1.8) with $\gamma > 1$ exists if and only if*

$$s_x(x,0) \ge 0 \quad \textit{and} \quad r_x(x,0) \ge 0, \quad \textit{for any} \quad x\,. \tag{2.22}$$

Remark 2.5. *At a point (x,t), the solution is said to be forward rarefactive (resp. compressive) if $s_x(x,t) \ge 0$ (resp. $s_x(x,t) < 0$); the solution is said to be backward rarefactive (resp. compressive) if $r_x(x,t) \ge 0$ (resp. $r_x(x,t) < 0$).*

Hence the theorem can be understood as that classical global-in-time solution of p-system exists if and only if the initial data are nowhere compressive.

If (2.22) is not satisfied, gradient blowup happens in finite time.

Proof. Step (1). Sufficiency. This part can be proved by standard local existence and global a priori C^1 estimates argument, c.f. [11], under the help of the lower bound of density provided in [14].

To make this paper self-contained, we give the sketch of proof. One can first prove the local-in-time existence of classical solutions for (1.6)∼(1.8), c.f. [11], where the life-span depends on the C^1-norm of u and τ and the positive lower bound on τ in the initial data. Then to extend the local-in-time existence of classical solution to global-in-time existence, we only have to get global a priori C^1 bounds on u and τ and the positive lower bound on τ for classical solutions. In fact, the upper bound on density and $|u|$ can be easily found by studying (2.16). Furthermore, Lemma 2.1 and (2.22) tell us that y and q are always nonnegative, while Lemma 2.3 gives us the upper bounds on y and q. Finally, the classical solution also satisfies the time-dependent density lower bound in [14] when (2.22) is satisfied by the weak-strong uniqueness, c.f. [8].

Step (2). Necessity. We only have to consider the case $1 < \gamma < 3$, in which a_2 does not have positive lower bound. In fact, it is enough to show that

$$\lim_{t\to\infty}\int_0^t a_2\, dt = \infty\,.$$

Via the definition of c in (2.12), (2.16) and Lemma 2.3, we have

$$s_t = -cs_x = -K_c\,\eta^{\frac{\gamma+1}{2(\gamma-1)}}\,y \geq -K_c\,\eta^{\frac{\gamma+1}{2(\gamma-1)}}\,Y\,,$$

and

$$r_t = cr_x = K_c\,\eta^{\frac{\gamma+1}{2(\gamma-1)}}\,q \leq K_c\,\eta^{\frac{\gamma+1}{2(\gamma-1)}}\,Q\,.$$

Hence

$$(s-r)_t \geq -K_c\,\eta^{\frac{\gamma+1}{2(\gamma-1)}}\,(Y+Q)\,,$$

so by (2.15),

$$\eta_t \geq -\frac{K_c}{2}\,\eta^{\frac{\gamma+1}{2(\gamma-1)}}\,(Y+Q)\,.$$

Dividing the above inequality by $\eta^{\frac{\gamma+1}{2(\gamma-1)}}$, then integrating both sides on t, we have

$$-\frac{2(\gamma-1)}{3-\gamma}(\eta(x,t))^{\frac{\gamma-3}{2(\gamma-1)}} + \frac{2(\gamma-1)}{3-\gamma}(\eta(x,0))^{\frac{\gamma-3}{2(\gamma-1)}} \geq -\frac{K_c}{2}(Y+Q)t\,.$$

Hence, when $1 < \gamma < 3$, using (2.10) we have

$$\tau(x,t) \leq \Big\{K_0\big[\tau^{\frac{3-\gamma}{4}}(x,0) + \frac{K_c}{2}(Y+Q)t\big]\Big\}^{\frac{4}{3-\gamma}}\,.$$

So

$$a_2 \geq K_{00}\cdot\Big\{K_0\big[\tau^{\frac{3-\gamma}{4}}(x,0) + \frac{K_c}{2}(Y+Q)t\big]\Big\}^{-1}\,.$$

Hence,

$$\int_0^{\infty} a_2 \, dt = \infty \,,$$

since $\tau^{\frac{3-\gamma}{4}}(x,0)$ and $Y+Q$ are bounded. So we prove singularity formation when $1<\gamma<3$. The proof of the theorem is completed. □

Remark 2.6. *Finally, we give a remark why the singularity in Theorem 2.4 is in fact a shock wave satisfying Lax entropy condition.*

First, at (x^*,t^*) where the first singularity formation happens, there are some characteristics in the same family interacting with each other. Let's prove it by contradiction. Assume there are no characteristics in the same family interacting with each other at or before time t^*, then C^1 solution exists when $t\in[0,t^*]$, which is contradict to the singularity formation at time t^*.

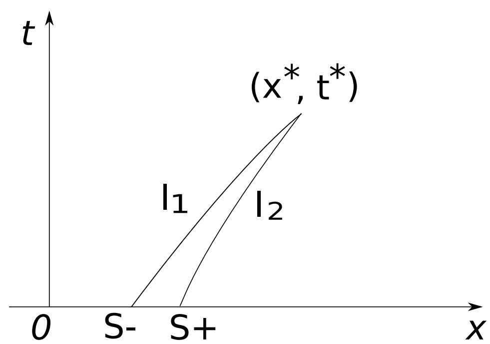

Figure 1: Shock formation

Hence, without loss of generality, we could find two characteristics l_1 and l_2 interacting with each other (see Figure 1) at (x^*,t^*), and $s_x<0$ in the region between l_1 and l_2 near (x^*,t^*) on (x,t)-plane, because $s_x\to-\infty$ when (x,t) approaches (x^*,t^*). Hence the solution is discontinuous when singularity forms because

$$\lim_{x\to x^*-} s(x,t^*) < \lim_{x\to x^*+} s(x,t^*).$$

Finally we check the Lax entropy condition. For smooth solutions before *blowup, by* (1.6)$\sim$(1.7) *and* (2.10)$\sim$(2.16)*,*

$$\begin{aligned} -cs_x &= s_t \\ &= u_t + \eta_t \\ &= -p_x + \eta_t \\ &= \partial_- \eta \,. \end{aligned}$$

So $\partial_- c \to +\infty$ *when* (x,t) *approaches* (x^*, t^*)*, hence the solution is discontinuous when singularity forms, and the Lax entropy condition is satisfied on the discontinuity.*

2.4 p-system with general pressure law

In this subsection, we consider the p-system (1.6)$\sim$(1.7) with general C^3 pressure $p(\tau)$ satisfying

$$p_\tau < 0, \quad p_{\tau\tau} > 0 \tag{2.23}$$

and

$$\lim_{\tau\to 0} p(\tau) = \infty, \quad \lim_{\tau\to\infty} p(\tau) = 0 \quad \text{and} \quad \int_1^\infty \sqrt{-p_\tau}\, d\tau < \infty \tag{2.24}$$

where condition (2.23) is dictated by physics when one uses (1.6)$\sim$(1.7) to model gas dynamics, c.f. [21]. Furthermore, we assume that

$$\int_0^1 \sqrt{-p_\tau}\, d\tau = \infty \tag{2.25}$$

which includes the γ-law pressure case.

Applying Lax's decomposition to this case we easily have the following proposition, where the detail calculation can be found in [4]

Proposition 2.7. *[4] The smooth solutions of* (1.6)$\sim$(1.7) *satisfy*

$$y' = -a(\tau) y^2, \tag{2.26}$$
$$q^{\backprime} = -a(\tau) q^2, \tag{2.27}$$

where

$$a(\tau) := \frac{p_{\tau\tau}}{4(-p_\tau)^{\frac{5}{4}}} > 0, \tag{2.28}$$

and

$$y := \sqrt{c}\, s_x, \qquad q := \sqrt{c}\, r_x, \tag{2.29}$$

with Lagrangian wave speed

$$c \equiv c(v) = \sqrt{-p_\tau},$$

and Riemann invariants

$$s := u + \int_\tau^1 c(\tau)d\tau \quad \textit{and} \quad r := u - \int_\tau^1 c(\tau)d\tau.$$

Furthermore,

$$s_t + cs_x = 0 \quad \textit{and} \quad r_t - cr_x = 0\,. \tag{2.30}$$

Secondly, we have the following lemma as same as Lemma (2.3).

Lemma 2.8. *For C^1 solutions of (1.6)-(1.7) and* (2.23), *we have* a priori *bounds*

$$y(x,t) \le \max\Big\{0,\ \sup_x\{y(x,0)\}\Big\} =: Y$$

$$\textit{and} \quad q(x,t) \le \max\Big\{0,\ \sup_x\{q(x,0)\}\Big\} =: Q\,.$$

Then we could state our theorem for the general pressure law case.

Theorem 2.9. *Assume that initial data $(u(x,0),\tau(x,0))$ are C^1, $|u(x,0)|$, $\tau(x,0)$, $|s_x(x,0)|$ and $|r_x(x,0)|$ are uniformly bounded above, and $\tau(x,0)$ is uniformly away from zero. The pressure satisfies* (2.23)$\sim$(2.25)*. Furthermore, assume there exists some positive constant A, such that for any $\tau > 0$,*

$$(5 + A)(p_{\tau\tau})^2 - 4p_\tau p_{\tau\tau\tau} \ge 0\,. \tag{2.31}$$

Then global-in-time classical solution of (1.6)$\sim$(1.7) exists if and only if

$$s_x(x,0) \ge 0 \quad \textit{and} \quad r_x(x,0) \ge 0, \quad \textit{for any} \quad x\,. \tag{2.32}$$

Remark 2.10. *It is clear that for the singularity formation, conditions* (2.24)$\sim$(2.25) *are not necessary.*

Condition (2.31) *is a fairly mild assumption because the constant A can be arbitrarily large. For example, the γ-law pressure $p = k\tau^{-\gamma}$ with $\gamma > 0$ satisfies conditions* (2.31) *and* (2.23)*, and the pressure $p = k\tau^{-\gamma}$ with $\gamma > 1$ satisfies conditions* (2.31) *and* (2.23)$\sim$(2.25)*.*

Proof. We first remark on a fact for later references. By (2.30), (2.25), it is easy to see that τ has a uniform lower bound. We denote that

$$\tau_{min} = \min_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} \tau(x,t)$$

which is a positive constant only depending on the maximum values of $|s(x,0)|$ and $|r(x,0)|$. Then it is easy to check that c has a uniform upper bound by (2.23).

If condition (2.32) holds, the global existence could be proved in an entirely similar way as we introduced in Theorem 2.4 together with the lower bound on density provided in [14].

If condition (2.32) fails, by a similar argument in Theorem 2.4, in order to prove singularity formation in finite time, we only have to show

$$\int_0^\infty a(\tau(x(t),t)\ dt = \infty\,. \tag{2.33}$$

Hence it is sufficient to show that

$$\frac{1}{a(\tau(x,t))} = \frac{4(-p_\tau)^{\frac{5}{4}}}{p_{\tau\tau}} \le K_1 + K_2 t \tag{2.34}$$

for some positive K_1 and K_2.

Then we prove (2.34). By Proposition 2.7, we have

$$\frac{1}{2}(y+q) = \frac{1}{2}\sqrt{c}(s_x + r_x) = \sqrt{c}\,u_x = \sqrt{c}\,\tau_t,$$

then by Lemma 2.8 we have

$$\Big(\int_{\tau_{min}}^{\tau} (-p_\tau(\tau))^{\frac{1}{4}}\ d\tau\Big)_t = \Big(\int_{\tau_{min}}^{\tau} \sqrt{c(\tau)}\ d\tau\Big)_t = \frac{1}{2}(y+q) \le \frac{1}{2}(Y+Q)$$

Hence

$$\int_{\tau_{min}}^{\tau(x,t)} (-p_\tau(\tau))^{\frac{1}{4}}\ d\tau \le \int_{\tau_{min}}^{\tau(x,0)} (-p_\tau(\tau))^{\frac{1}{4}}\ d\tau + \frac{1}{2}(Y+Q)t \le K_3 + K_4 t \tag{2.35}$$

for some positive constants K_3 and K_4.

Compare (2.34) and (2.35), and use $\tau > \tau_{min} > 0$, then it is easy to see that in order to get (2.34) we only have to show that

$$\Big(\frac{4(-p_\tau)^{\frac{5}{4}}}{p_{\tau\tau}}\Big)_\tau \le A(-p_\tau(\tau))^{\frac{1}{4}}, \tag{2.36}$$

for some positive constant A. In fact, inequality (2.36) means that the left hand side of (2.34) grows slower than the left hand side of (2.35) as τ increases. It is easy to see that (2.36) is correct because of (2.31). Hence we finish the proof of this Theorem. $\square$

3 Compressible Euler equations

In this section, we consider the Cauchy problem of compressible Euler equations (1.1)∼(1.4) with given smooth initial data $(u(x,0),\tau(x,0),S(x,0))$. We will establish a time-dependent lower bound on density for smooth solutions, then prove the main singularity formation results: Theorems 3.5 and 3.10.

We will provide our result through several steps. We first introduce the coordinates and equations which are basically from [2], then review the uniform upper bound on density and velocity established in [6], and finally provide the lower bound on density and prove Theorem 3.5 when initial entropy has finite total variation and Theorem 3.10 when entropy has infinite total variation.

Finally, in Subsection 3.5, we explain why the condition on initial data for singularity formation in our Theorems are the best we could expect.

3.1 Equations and coordinates

We use the coordinates used in [2]. And all detail calculation in this subsection can be found in [2] or [6]. Define new variables (m,η) for (S,τ), by

$$m := e^{\frac{S}{2c_v}} > 0 \tag{3.1}$$

and

$$\eta := \int_\tau^\infty \frac{c}{m}\,d\tau = \tfrac{2\sqrt{K\gamma}}{\gamma-1}\,\tau^{-\frac{\gamma-1}{2}} > 0\,, \tag{3.2}$$

where the nonlinear Lagrangian sound speed c is

$$c := \sqrt{-p_\tau} = \sqrt{K\,\gamma}\,\tau^{-\frac{\gamma+1}{2}}\,e^{\frac{S}{2c_v}}\,. \tag{3.3}$$

Without confusions, we still use η and c for full Euler equations. In fact, functions η and c in p-system equals to η and c for full Euler equations when $m=1$, respectively. Similarly, in this section, if we use the same letter as in p-system to denote a function, this function is extended form the corresponding function in p-system, which should have no ambiguity.

It follows that

$$\begin{aligned} \tau &= K_\tau\,\eta^{-\frac{2}{\gamma-1}}\,, \\ p &= K_p\,m^2\,\eta^{\frac{2\gamma}{\gamma-1}}\,, \\ c &= c(\eta,m) = K_c\,m\,\eta^{\frac{\gamma+1}{\gamma-1}}\,. \end{aligned} \tag{3.4}$$

In these coordinates, for C^1 solutions, equations (1.1)–(1.3) are equivalent to

$$\eta_t + \frac{c}{m}\, u_x = 0\,, \tag{3.5}$$

$$u_t + m\, c\, \eta_x + 2\frac{p}{m}\, m_x = 0\,, \tag{3.6}$$

$$m_t = 0\,, \tag{3.7}$$

where the last equation comes from (1.5), which is equivalent to (1.3), c.f. [8, 24]. Note that, while the solution remains C^1, $m = m(x)$ is given by the initial data and can be regarded as a stationary quantity.

We denote the Riemann invariants by

$$r := u - m\,\eta\,, \qquad s := u + m\,\eta\,. \tag{3.8}$$

Different from the isentropic case (m constant), for general non-isentropic flow, s and r vary along characteristics.

The forward and backward characteristics are described by

$$\frac{dx}{dt} = c \quad \text{and} \quad \frac{dx}{dt} = -c\,, \tag{3.9}$$

and we denote the corresponding directional derivatives along these by

$$\partial_+ := \tfrac{\partial}{\partial t} + c\,\tfrac{\partial}{\partial x} \quad \text{and} \quad \partial_- := \tfrac{\partial}{\partial t} - c\,\tfrac{\partial}{\partial x}\,,$$

respectively. Using (3.8), (3.4) and (2.14), equations (3.5) and (3.6) give

$$\partial_+ s = \frac{1}{2\gamma}\,\frac{c\, m_x}{m}\,(s-r)\,, \tag{3.10}$$

$$\partial_- r = \frac{1}{2\gamma}\,\frac{c\, m_x}{m}\,(s-r)\,. \tag{3.11}$$

Then we introduce gradient variables

$$\begin{aligned} y &:= m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}\, \eta^{\frac{\gamma+1}{2(\gamma-1)}}\, \big((u+m\,\eta)_x - \tfrac{2}{3\gamma-1}\, m_x\, \eta\big) \quad \text{and} \\ q &:= m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}\, \eta^{\frac{\gamma+1}{2(\gamma-1)}}\, \big((u-m\,\eta)_x + \tfrac{2}{3\gamma-1}\, m_x\, \eta\big)\,, \end{aligned} \tag{3.12}$$

and derive Riccati type equations for their evolution:

Lemma 3.1. *[2] For C^1 solutions of (1.1)∼(1.4), we have*

$$\begin{aligned} \partial_+ y &= a_0 - a_2\, y^2, \\ \partial_- q &= a_0 - a_2\, q^2, \end{aligned} \tag{3.13}$$

where

$$a_0 := \tfrac{K_c}{\gamma}\left[\tfrac{\gamma-1}{3\gamma-1}\, m\, m_{xx} - \tfrac{(3\gamma+1)(\gamma-1)}{(3\gamma-1)^2}\, m_x^2\right] m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}\, \eta^{\frac{3(\gamma+1)}{2(\gamma-1)}+1},$$
$$a_2 := K_c\, \tfrac{\gamma+1}{2(\gamma-1)}\, m^{\frac{3(3-\gamma)}{2(3\gamma-1)}}\, \eta^{\frac{3-\gamma}{2(\gamma-1)}}. \tag{3.14}$$

Furthermore,

$$|y| \text{ or } |q| \to \infty \quad \text{iff} \quad |u_x| \text{ or } |\tau_x| \to \infty\,. \tag{3.15}$$

The proof of this lemma can be found in [2]. The decomposition (3.13), which is generalized from the Lax's decomposition in [10] for hyperbolic system with two unknowns, was first provided by [17], and then found in [2] independently through another explanation.

3.2 Uniform upper bound on density

In this part, we review a result on the uniform upper bounds for $|u|$ and ρ provided by G. Chen, R. Young and Q. Zhang in [6], for later references.

Assume that the initial entropy $S(x)$ is C^1 and has finite total variation, so that

$$V := \frac{1}{2c_v}\int_{-\infty}^{+\infty} |S'(x)|\, dx = \int_{-\infty}^{+\infty} \frac{|m'(x)|}{m(x)}\, dx < \infty\,, \tag{3.16}$$

while also, by (3.1),

$$0 < M_L < m(\cdot) < M_U\,, \tag{3.17}$$

for some constants M_L and M_U. Also, we assume that $\rho > 0$ and $|u|$ are bounded above initially. Hence, there exist positive constants M_s and M_r, such that, in the initial data,

$$|s_0(\cdot)| < M_s \quad \text{and} \quad |r_0(\cdot)| < M_r\,. \tag{3.18}$$

In this section, we always assume (3.16)$\sim$(3.18).

We define two useful constants by

$$N_1 := M_s + \overline{V}\, M_r + \overline{V}\,(\overline{V}\, M_s + \overline{V}^2\, M_r)\, e^{\overline{V}^2},$$
$$N_2 := M_r + \overline{V}\, M_s + \overline{V}\,(\overline{V}\, M_r + \overline{V}^2\, M_s)\, e^{\overline{V}^2},$$

where

$$\overline{V} := \frac{V}{2\gamma}\,,$$

which clearly depend only on the initial data. By below proposition in [6], $|u|$ and ρ are shown to be uniformly bounded above.

Proposition 3.2. [6] *Assume system* (1.1)$\sim$(1.4), *with provided initial data satisfying* (3.16)$\sim$(3.18), *has a* C^1 *solution when* $t \in [0, T)$, *then one has the uniform bounds*

$$|u(x,t)| \leq \frac{N_1 + N_2}{2} M_U{}^{\frac{1}{2\gamma}} \quad \textit{and} \quad \rho(x,t) \leq \frac{N_1 + N_2}{2} M_L{}^{\frac{1}{2\gamma}-1}, \tag{3.19}$$

where T *can be any positive number or infinity. And the bounds are independent of* T.

3.3 Singularity formation when entropy has finite total variation

For full Euler equations, the major difficulty we need to overcome in this paper is still how to find the time dependent lower bound on density when $1 < \gamma < 3$. We first give a lemma to define a positive constant N for later use.

Lemma 3.3. *Assume the initial data* $u(x,0)$ *and* $\tau(x,0)$ *are* C^1 *and uniformly bounded above, the initial entropy* $S(x,0)$ *is* C^2 *and has bounded variation. Furthermore, suppose there is a positive constant* M_* *such that the initial entropy satisfies* $|m''(x)| < M_*$. *Then, for* C^1 *solution of system* (1.1)$\sim$(1.4), *there exists a positive constant* N *depending only on the initial data, such that*

$$\sqrt{\frac{a_0}{a_2}} \leq N \quad \textit{if} \quad a_0 \geq 0\,. \tag{3.20}$$

Furthermore, if y *or* q *is larger than* N *or less than* $-N$ *at* (x,t), *then*

$$\partial_+ y = a_0 - a_2\, y^2 < 0 \quad \textit{or} \quad \partial_- q = a_0 - a_2\, q^2 < 0 \quad \textit{at} \quad (x,t)\,, \tag{3.21}$$

respectively.

Proof. Clearly, the assumptions in Proposition 3.2 are also satisfied, by the assumptions in this lemma. Then by Proposition 3.2 and (3.2), we know η has a global upper bound depending only on the initial data, denoted via E_U.

By (3.14), it is easy to calculate that, if $a_0 \geq 0$,

$$\sqrt{\frac{a_0}{a_2}} = \sqrt{\tfrac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)} \left(m\, m_{xx} - \tfrac{3\gamma+1}{3\gamma-1}\, m_x^2\right)}\, \eta^{\frac{\gamma+1}{2(\gamma-1)}+1}\, m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}\,, \tag{3.22}$$

which implies the uniform bound $\sqrt{a_0/a_2} \le N$, where

$$N := \begin{cases} \sqrt{\frac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}\, M_*}\; E_U^{\frac{3\gamma-1}{2(\gamma-1)}}\; M_L^{\frac{3\gamma-5}{3\gamma-1}}, & 1<\gamma\le 5/3\,, \\ \sqrt{\frac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}\, M_*}\; E_U^{\frac{3\gamma-1}{2(\gamma-1)}}\; M_U^{\frac{3\gamma-5}{3\gamma-1}}, & \gamma\ge 5/3\,, \end{cases} \tag{3.23}$$

where M_L and M_U are defined in (3.17).

By (3.20) and (3.13), (3.21) is clearly correct. □

Next we prove a key lemma.

Lemma 3.4. *For C^1 solutions of (1.1)∼(1.4) with initial data satisfying (3.16)∼(3.18) and $|m_{xx}|$ is uniformly bounded above, we have*

$$y(x,t) \le \max\Big\{N,\ \sup_x\{y(x,0)\}\Big\} =: \bar{Y}\,,$$

$$\textit{and}\quad q(x,t) \le \max\Big\{N,\ \sup_x\{q(x,0)\}\Big\} =: \bar{Q}\,.$$

Proof. Without loss of generality, we only prove the inequality for y. Then the inequality for q will be proved in an entirely same way.

We prove the inequality for y by contradiction. Assume that

$$y(x^*,\,t^*) > \max\Big\{N,\ \sup_x\{y(x,0)\}\Big\}\,. \tag{3.24}$$

We use $\Gamma(t)$ with $t\in[\bar{t},\,t^*]$ to denote the largest connected piece of forward characteristic $x^+(t)$ containing $(x^*,\,t^*)$ as its upper endpoint, such that

$$y(x^+(t),\,t) \ge y(x^*,\,t^*)$$

for any points $(x^+(t),\,t)$ on $\Gamma(t)$.

Since

$$y(x^+(0),\,0) \le \max\Big\{N,\ \sup_x\{y(x,0)\}\Big\} < y(x^*,\,t^*)\,,$$

hence

$$\bar{t} > 0\,,$$

then by the definition of $\Gamma(t)$, we have

$$y(x^+(\bar{t}\,),\,\bar{t}\,) = y(x^*,\,t^*)\,.$$

On the other hand, by the note before this Lemma, we know

$$\frac{d}{dt} y(x^+(t),\, t) < 0, \quad \text{when} \quad t = t^*,$$

hence $\bar{t} < t^*$ and there exists some $\hat{t} \in (\bar{t},\, t^*)$ such that

$$y(x^+(\hat{t}\,),\, \hat{t}\,) > y(x^*,\, t^*)\,.$$

So

$$y(x^+(\bar{t}\,),\, \bar{t}\,) = y(x^*,\, t^*) < y(x^+(\hat{t}\,),\, \hat{t}\,)\,.$$

However, this is impossible because y is decreasing on t on Γ with $t \in [\bar{t},\, t^*]$ by (3.21) and (3.24).

Hence we find a contradiction. So the lemma is proved. □

This lemma is consistent with Lemma 2.3 for p-system. In fact, in p-system, $N = 0$.

Finally we prove our main theorem for full compressible Euler equations.

Theorem 3.5. *Assume that the initial data $u(x,0)$ and $\tau(x,0)$ are C^1, $|u(x,0)|$, $\tau(x,0)$, $y(x,0)$ and $q(x,0)$ are uniformly bounded above, $\tau(x,0)$ is uniformly away from zero, and $S(x,0)$ is C^2 and has bounded variation. Furthermore, suppose there is a positive constant M_* such that the initial entropy satisfies $|m''(x)| < M_*$. Then, for system* (1.1)∼(1.4)*, there exists a positive constant N defined in* (3.23) *which depends only on the initial data, such that, if the initial data satisfy*

$$\inf \Big\{ \; y(\cdot, 0), \;\; q(\cdot, 0) \; \Big\} < -N\,, \tag{3.25}$$

then $|u_x|$ and/or $|\tau_x|$ blow up in finite time.

Remark 3.6. *When $\gamma \geq 3$, this theorem has already been proved in [6]. To make the paper self-contained, we will still give the proof of this case.*

Proof. Suppose that (3.25) holds. Without loss of generality, we can assume that $\inf y < -N$. In fact the case when $\inf q < -N$ can be proved in an entirely same way. Then there exist $\varepsilon > 0$ and x_0 such that

$$y(x_0, 0) < -(1+\varepsilon)\, N\,. \tag{3.26}$$

Now considering the solution $y(t)$ on the forward characteristic $\Gamma_{x_0}(t)$ starting at $(x_0, 0)$. Along this characteristic $\Gamma_{x_0}(t)$, by Lemma 3.3 and recall (3.26), we have

$$\partial_+ y(t) < 0 \quad \text{and} \quad y(t) < -(1+\varepsilon)\,N \quad \text{for any} \quad t \geq 0\,,$$

which together with (3.20) implies that for all $t \geq 0$,

$$a_0 - a_2\,\frac{y^2(t)}{(1+\varepsilon)^2} < 0\,,$$

hence by (3.13) and $a_2 > 0$,

$$\partial_+ y(t) = a_0 - a_2\,y^2(t) < \Big(-1 + \tfrac{1}{(1+\varepsilon)^2}\Big)\,a_2\,y^2(t) < 0\,.$$

Integrating it, we get

$$\frac{1}{y(t)} \geq \frac{1}{y(0)} + \int_0^t \Big(1 - \tfrac{1}{(1+\varepsilon)^2}\Big)\,a_2\;dt\,, \tag{3.27}$$

where the integral is along the forward characteristic.

<u>Case when $\gamma \geq 3$</u>. By (3.14) and Proposition 3.2, a_2 is positive and bounded above, so the right hand side of (3.27) approaches zero in finite time. This implies that $y(t)$ approaches $-\infty$ in finite time, so that $|\tau_x|$ and/or $|u_x|$ blow up. This is a known result in [6] as discussed in Remark 3.6.

<u>Case when $1 < \gamma < 3$</u>. To prove the singularity formation in finite time, we only have to show

$$\int_0^\infty a_2\;dt = \infty\,,$$

by establishing a similar time dependent lower bound on density as the one for p-system. In fact, by (3.10)$\sim$(3.11) and Lemma 3.4,

$$\begin{aligned}
s_t &= -c\,s_x + \frac{1}{2\gamma}\frac{c\,m_x}{m}\,(s-r) \\
&= -K_c m^{\frac{3\gamma+7}{2(3\gamma-1)}}\eta^{\frac{\gamma+1}{2(\gamma-1)}}y - K_c\frac{2}{3\gamma-1}m_x m\eta^{\frac{2\gamma}{\gamma-1}} + \frac{1}{2\gamma}\frac{c\,m_x}{m}\,(s-r) \\
&\geq -K_2\eta^{\frac{\gamma+1}{2(\gamma-1)}}\bar{Y} - K_3\eta^{\frac{\gamma+1}{2(\gamma-1)}} + \frac{1}{2\gamma}\frac{c\,m_x}{m}\,(s-r)\,,
\end{aligned}$$

where to get positive constants K_2 and K_3, we need to use the uniform upper bound on density and bounds on initial entropy. Similarly

$$\begin{aligned}
r_t &= c\,r_x + \frac{1}{2\gamma}\frac{c\,m_x}{m}(s-r) \\
&= K_c m^{\frac{3\gamma+7}{2(3\gamma-1)}}\eta^{\frac{\gamma+1}{2(\gamma-1)}} q - K_c \frac{2}{3\gamma-1} m_x m \eta^{\frac{2\gamma}{\gamma-1}} + \frac{1}{2\gamma}\frac{c\,m_x}{m}(s-r) \\
&\leq K_2 \eta^{\frac{\gamma+1}{2(\gamma-1)}} \bar{Q} + K_3 \eta^{\frac{\gamma+1}{2(\gamma-1)}} + \frac{1}{2\gamma}\frac{c\,m_x}{m}(s-r)\,.
\end{aligned}$$

Clearly, $\bar{Y}$ and $\bar{Q}$ are both finite constants. Then we have

$$2m\,\eta_t = (s-r)_t \geq -[K_2(\bar{Y}+\bar{Q}) + 2K_3]\,\eta^{\frac{\gamma+1}{2(\gamma-1)}}.$$

So

$$\eta_t \geq -K_4[K_2(\bar{Y}+\bar{Q}) + 2K_3]\,\eta^{\frac{\gamma+1}{2(\gamma-1)}}$$

because m defined in (3.1) has positive lower bound. Similar as in p-system, we have when $1<\gamma<3$,

$$\tau(x,t) \leq \Big\{K_5\Big[\tau^{\frac{3-\gamma}{4}}(x,0) + K_4\big(K_2(\bar{Y}+\bar{Q}) + 2K_3\big)t\Big]\Big\}^{\frac{4}{3-\gamma}}.$$

Hence

$$a_2 \geq K_6\Big\{\tau^{\frac{3-\gamma}{4}}(x,0) + K_4\big(K_2(\bar{Y}+\bar{Q}) + 2K_3\big)t\Big\}^{-1}, \qquad (3.28)$$

which gives

$$\lim_{t\to\infty}\int_0^t a_2\,dt = \infty\,.$$

Hence, singularity forms in finite time.

□

Corollary 3.7. *Suppose all assumptions in Theorem 3.5 hold. Then when* $1<\gamma<3$*,*

$$\tau(x,t) \leq \Big\{K_5\Big[\tau^{\frac{3-\gamma}{4}}(x,0) + K_4\big(K_2(\bar{Y}+\bar{Q}) + 2K_3\big)t\Big]\Big\}^{\frac{4}{3-\gamma}}.$$

See the definitions of positive constants of $K_2 \sim K_5$*,* $\bar{Y}$ *and* $\bar{Q}$ *in the proof of Theorem 3.5.*

3.4 Singularity formation when entropy has infinite total variation

In this section, we consider the singularity formation when the entropy has infinite total variation on the whole real line but finite total variation on any bounded interval, i.e. entropy is local BV, which includes interesting cases such as periodic solutions.

Since when we prove (3.19) for $\rho(x,t)$ and $|u(x,t)|$ in [6], we only have to consider the domain of dependence of the point (x,t). Hence, we can get similar upper bounds on density and velocity as those in (3.19) in below remark on a domain of dependence, when initial entropy is only local BV.

Remark 3.8. *Consider the domain of dependence of a point $(X_{x,z}, T_{x,z})$, in Figure 2, which is denoted by $\Omega_{x,z}$, where $[x,z]$ is the intersection interval of the domain of dependence and the initial line $t=0$.*

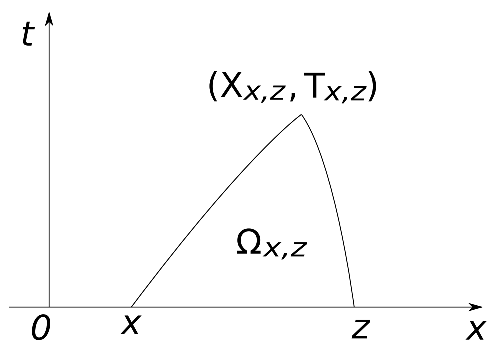

Figure 2: A domain of dependence $\Omega_{x,z}$

We still assume (3.17)$\sim$(3.18). *But instead of (3.16), we suppose that S is C^1 and local BV, which shows*

$$V_{x,z} := \frac{1}{2c_\tau}\int_x^z |S'(\bar{x})|\, d\bar{x} = \int_x^z \frac{|m'(\bar{x})|}{m(\bar{x})}\, d\bar{x} < \infty\,. \tag{3.29}$$

By the same proof in [6], it is easy to get, for any point $(\bar{x},t)$ in $\Omega_{x,z}$

$$|u(\bar{x},t)| \le \frac{N_{1x,z}+N_{2x,z}}{2} M_U{}^{\frac{1}{2\gamma}} \quad \textit{and} \quad \rho(\bar{x},t) \le \frac{N_{1x,z}+N_{2x,z}}{2} M_L{}^{\frac{1}{2\gamma}-1} \tag{3.30}$$

with

$$N_{1x,z} := M_s + \bar{V}_{x,z}\, M_r + \bar{V}_{x,z}\,(\bar{V}_{x,z}\, M_s + \bar{V}_{x,z}^2\, M_r)\, e^{\bar{V}_{x,z}^2},$$
$$N_{2x,z} := M_r + \bar{V}_{x,z}\, M_s + \bar{V}_{x,z}\,(\bar{V}_{x,z}\, M_r + \bar{V}_{x,z}^2\, M_s)\, e^{\bar{V}_{x,z}^2},$$

where

$$\bar{V}_{x,z} := \frac{V_{x,z}}{2\gamma}\,.$$

Remark 3.9. *Still under the assumptions and notations in Remark 3.8, we introduce some new notations for later reference. We denote the maximum density value in $\Omega_{x,z}$ to be $\rho_{x,z}$ which satisfies*

$$\rho_{x,z} \le \frac{N_{1x,z} + N_{2x,z}}{2} M_L{}^{\frac{1}{2\gamma}-1}.$$

Denote the maximum η value in $\Omega_{x,z}$ to be $E_{U\,x,z}$ which satisfies

$$E_{U\,x,z} \le \frac{2\sqrt{K\gamma}}{\gamma-1}\,\rho_{x,z}^{\frac{\gamma-1}{2}} \tag{3.31}$$

by (3.2)*. Denote the time at the upper vertex of $\Omega_{x,z}$ as $T_{x,z}$ which satisfies*

$$\frac{1}{T_{x,z}} \le \frac{2}{z-x}\cdot K_c\, E_{U\,x,z}^{\frac{\gamma+1}{\gamma-1}}\, M_U,$$

by (3.9)*. And if further assume that $m''(\bar{x}) < M_*$ for any $\bar{x} \in [x,z]$, then*

$$\left(\sqrt{\tfrac{|a_0|}{a_2}}\right)_{x,z} \le N_{x,z} := \begin{cases} \sqrt{\frac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}\,M_*}\; E_{U\,x,z}^{\frac{3\gamma-1}{2(\gamma-1)}}\; M_L^{\frac{3\gamma-5}{3\gamma-1}}, & 1<\gamma\le\frac{5}{3}\,,\\ \sqrt{\frac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}\,M_*}\; E_{U\,x,z}^{\frac{3\gamma-1}{2(\gamma-1)}}\; M_U^{\frac{3\gamma-5}{3\gamma-1}}, & \gamma\ge\frac{5}{3}\,, \end{cases} \tag{3.32}$$

where $\left(\sqrt{\frac{|a_0|}{a_2}}\right)_{x,z}$ denotes the maximum value of $\sqrt{\frac{|a_0|}{a_2}}$ in $\Omega_{x,z}$ when $a_0 \ge 0$. If a_0 is always negative in $\Omega_{x,z}$, define $\left(\sqrt{\frac{|a_0|}{a_2}}\right)_{x,z} = 0$.

When $1<\gamma<3$, assuming initial density has uniform lower bound, by a similar argument as for (3.28)*, we have in $\Omega_{x,z}$*

$$a_2 \ge K_{7x,z}(1 + K_{8x,z}t)^{-1}\,, \tag{3.33}$$

where $K_{7x,z} > 0$ and $K_{8x,z} > 0$ depend on the initial bounds in (3.17)$\sim$(3.18)*, lower bound on initial density and initial bound $V_{x,z}$ defined in* (3.29) *for entropy.*

Theorem 3.10. *Assume the initial data $u(x,0)$ and $\tau(x,0)$ are C^1, $|u(x,0)|$, $\tau(x,0)$, $y(x,0)$ and $q(x,0)$ are uniformly bounded above, $\tau(x,0)$ is uniformly away from zero, and $S(x,0)$ is C^2 and local BV. Furthermore, suppose there is a positive constant M_* such that the initial entropy satisfies $|m''(x)| < M_*$.*

Then, for system (1.1)$\sim$(1.4)*, if there exists some interval* (x,z) *such that the initial data satisfy*

$$y(x,0) \leq -N_{x,z}(1+B_{x,z})\,, \tag{3.34}$$

where $N_{x,z}$ *is defined in* (3.32) *and* $B_{x,z}$ *satisfies*

$$\frac{B_{x,z}(2+B_{x,z})}{(1+B_{x,z})} > \begin{cases} \left(\frac{K_c(\gamma+1)}{2(\gamma-1)}\,N_{x,z}T_{x,z}\,M_U^{\frac{3(3-\gamma)}{2(3\gamma-1)}}\,E_{U\,x,z}^{\frac{3-\gamma}{2(\gamma-1)}}\right)^{-1}, & \gamma \geq 3\,, \\ \left(\frac{K_{7x,z}}{K_{8x,z}}\,N_{x,z}\,\ln(1+K_{8x,z}T_{x,z})\right)^{-1}, & 1<\gamma<3\,, \end{cases} \tag{3.35}$$

then $|u_x|$ *and/or* $|\tau_x|$ *blow up in finite time. See Remarks 3.8, 3.9 and Figure 2 for definitions of notations. Symmetric result holds in the backward direction for* q*.*

Remark 3.11. *The right hand side of* (3.35) *only depends on the initial data. For any given local BV initial entropy, condition* (3.35) *will be satisfied when* $B_{x,z}$ *is large enough, i.e.* $y(x,0)$ *is negative enough. This means that singularity forms in finite time when the initial compression is strong enough somewhere.*

One sufficient condition on $B_{x,z}$ *such that (3.35) is satisfied is that*

$$B_{x,z} > \begin{cases} \left(\frac{K_c(\gamma+1)}{2(\gamma-1)}\,N_{x,z}T_{x,z}\,M_U^{\frac{3(3-\gamma)}{2(3\gamma-1)}}\,E_{U\,x,z}^{\frac{3-\gamma}{2(\gamma-1)}}\right)^{-1}, & \gamma \geq 3\,, \\ \left(\frac{K_{7x,z}}{K_{8x,z}}\,N_{x,z}\,\ln(1+K_{8x,z}T_{x,z})\right)^{-1}, & 1<\gamma<3\,, \end{cases} \tag{3.36}$$

This result is consistent with Theorem 3.5. In fact, when the initial entropy has finite total variation, $T_{x,\infty}=\infty$ *while* $N_{x,\infty}$*,* $K_{7x,\infty}$ *and* $K_{8x,\infty}$ *are all finite, so* $B_{x,\infty}$ *can be arbitrarily small. Hence, if* $y(x,0)<-N_{x,\infty}$*, then blowup happens in finite time.*

Proof. We only consider the solution in the domain of dependence $\Omega_{x,z}$, and prove that singularity formation happens in $\Omega_{x,z}$. More precisely, we will show that y goes to negative infinity along the forward characteristic starting from the point $(x,0)$ before $T_{x,z}$. We still use $y(t)$ to denote this characteristic.

From now on, we restrict our consideration in $\Omega_{x,z}$. Then we can use all bounds in Remark 3.9. By (3.13), (3.32) and (3.34), we know

$$\partial_+ y(t) < 0 \quad \text{and} \quad y(t) \leq -N_{x,z}(1+B_{x,z}), \quad \text{for any} \quad 0<t<T_{x,z}.$$

Still by (3.32), when $0 < t < T_{x,z}$,

$$a_0 - a_2 \frac{y^2(t)}{(1+B_{x,z})^2} < 0\,,$$

so

$$\partial_+ y(t) = a_0 - a_2\, y^2(t) < -\frac{B_{x,z}(2+B_{x,z})}{(1+B_{x,z})^2}\, a_2\, y^2(t) < 0\,.$$

Integrating it, we get

$$\frac{1}{y(t)} \geq \frac{1}{y(0)} + \frac{B_{x,z}(2+B_{x,z})}{(1+B_{x,z})^2} \int_0^t a_2\, dt\,. \tag{3.37}$$

Hence the blowup happens when the right hand side of (3.37) equals to zero, i.e. when

$$-\frac{1}{y(0)} = \frac{B_{x,z}(2+B_{x,z})}{(1+B_{x,z})^2} \int_0^t a_2\, dt\,. \tag{3.38}$$

Now to complete the proof of the Theorem 3.10, the only thing left to show is that blowup happens before $T_{x,z}$. By (3.34) we only have to show that

$$\frac{1}{N_{x,z}} \leq \frac{B_{x,z}(2+B_{x,z})}{(1+B_{x,z})} \int_0^{T_{x,z}} a_2\, dt\,. \tag{3.39}$$

Finally, we prove (3.39) case by case.

Case when $\gamma \geq 3$. By (3.31), in $\Omega_{x,z}$,

$$a_2 \geq K_c{}_{\frac{\gamma+1}{2(\gamma-1)}}\, M_U{}^{\frac{3(3-\gamma)}{2(3\gamma-1)}}\, E_U{}_{x,z}^{\frac{3-\gamma}{2(\gamma-1)}}\,.$$

Hence by (3.35), clearly (3.39) is satisfied.

Case when $1 < \gamma < 3$. By (3.33), we know to prove (3.39), we only have to show that

$$\frac{1}{N_{x,z}} \leq \frac{B_{x,z}(2+B_{x,z})}{(1+B_{x,z})} \int_0^{T_{x,z}} K_{7x,z}(1+K_{8x,z}t)^{-1}\, dt\,, \tag{3.40}$$

which is clearly correct by (3.35).

Hence we complete the proof of this theorem.

$\square$

3.5 Further discussion

Finally we give an example to show the sharpness of the Theorem 3.5.

We consider a global smooth stationary solution

$$u = 0, \quad m = m(x) \quad \text{and} \quad \tau = \tau(x)$$

which satisfies

$$p_x(x) = 0. \tag{3.41}$$

Then we provide the profile of $m(x)$, satisfiing all conditions in Theorem 3.5, by which we can get $\tau(x)$ using (3.41) and (3.4).

First, by (3.41) and (3.4),

$$-q(x) = y(x) = \tfrac{\gamma-1}{\gamma(3\gamma-1)} m_x m^{\frac{3(\gamma-3)}{2(3\gamma-1)}} \eta^{\frac{\gamma+1}{2(\gamma-1)}+1}\,. \tag{3.42}$$

We note that N is the best estimate we could have now for the upper bound of

$$\sqrt{\tfrac{a_0}{a_2}} = \sqrt{\tfrac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}\left(m\, m_{xx} - \tfrac{3\gamma+1}{3\gamma-1}\, m_x^2\right)}\, \eta^{\frac{\gamma+1}{2(\gamma-1)}+1}\, m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}\,, \tag{3.43}$$

by (3.22), when $m\, m_{xx} - \frac{3\gamma+1}{3\gamma-1}\, m_x^2 \geq 0$.

Comparing (3.42) with (3.43), we see if

$$\tfrac{\gamma-1}{\gamma(3\gamma-1)}|m_x| = \sqrt{\tfrac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}\left(m\, m_{xx} - \tfrac{3\gamma+1}{3\gamma-1}\, m_x^2\right)}$$

which is equivalent to

$$\tfrac{6\gamma^2+3\gamma+1}{\gamma(3\gamma-1)} m_x^2 = 2 m m_{xx}$$

or

$$S_{xx} = \tfrac{5\gamma+1}{4c_v\gamma(3\gamma-1)} S_x^2\,, \tag{3.44}$$

where S is the entropy satisfying (3.1), then either $y(x) = -\sqrt{\frac{a_0(x)}{a_2(x)}}$ or $q(x) = -\sqrt{\frac{a_0(x)}{a_2(x)}}$.

It is clear that we could find a positive solution $S(x)$ of (3.44) in the region $x \in [1, 2]$, once we set $S(1)$ to be a number large enough. Then choose the function $S(x)$ to be almost constant in either $(-\infty, 1)$ or $(2, \infty)$. Finally we get $\tau(x)$ by (3.41) and (3.4). And it is easy to see that this solution is a global smooth solution.

Now we already found a smooth stationary solution, satisfying either $y(x) = -\sqrt{\frac{a_0(x)}{a_2(x)}}$ or $q(x) = -\sqrt{\frac{a_0(x)}{a_2(x)}}$ for each $x \in [1, 2]$. On the other hand,

$y(x)$, $q(x)$ and $\sqrt{\frac{a_0(x)}{a_2(x)}}$ are all almost zero in the set outside $x \in (1-\varepsilon, 2+\varepsilon)$, for a very small constant ε, by (3.42)∼(3.43) and m_x is almost zero in that set. In conclusion,

$$\inf\,(\,y,\,q\,) \approx -\min_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} \sqrt{\frac{a_0}{a_2}}.$$

Till now, $-N$, provided by the initial data, is the best estimate for $-\min_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} \sqrt{\frac{a_0}{a_2}}$. If one could not improve this estimate, then the condition (3.25) provides a sharp condition, under which classical solution must break down in finite time.

Finally, when the maximum oscillation of initial entropy approaches zero, M_* approaches zero, then N and $N_{x,z}$ tend to zero. When m is constant, $N = 0$ and $N_{x,z} = 0$. Hence, Theorems 3.5 and 3.10 for non-isentropic Euler equations are consistent with Theorem 2.4 for p-system: when the initial entropy oscillation is weak enough, classical solution must break down in finite time even there are only weak initial compressions.

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